

THE PENMAN-MONTEITH SYSTEM OF EQUATIONS: Derivation with the Thermodynamic Approach, Numerical Solutions, and Evaluation

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1. Executive Summary

A review of the derivation of the Penman-Monteith equation with the thermodynamic approach of Monteith (1981) is presented in this document. Evapotranspiration (henceforth referred to simply as evaporation) is described here as a process consisting of a pair of formal thermodynamic subprocesses (namely, adiabatic cooling and diabatic heating) that leads to changes in the energy states of the ambient air in a way that is readily quantifiable. The thermodynamic formulation is used in Chapter 2 to derive the Penman-Monteith equation in a pair of steps. As an initial approximation, first a form of the equation that models evaporation into a stationary ambient air, from a wet source/sink surface, is developed based on thermodynamic equations of state applied to a suitably defined system. In a subsequent step, transfer coefficients (resistance parameters) are introduced into the basic equations accounting for the dynamic effects of wind-surface interactions (and the effects of crop physiological response to atmospheric conditions) on evaporation, leading to the Penman-Monteith equation.

Although less compact than the conventional approach (Penman, 1948; Jensen and Allen, 2016), the thermodynamic approach to the derivation of the Penman-Monteith equation has the advantage of being revealing of some key assumptions and concepts that are generally implicit in the conventional approach. The initial step of the thermodynamic formulation accentuates the notion that the Penman-Monteith equation is fundamentally a description of the process of vapor and heat transfer between a wet source/sink surface and a stationary ambient air. The subsequent step, on the other hand, emphasizes the fact that wind and crop effects on evaporation are taken into account in an approximate sense through the introduction of resistance parameters. The thermodynamic approach also shows that evaporation is essentially a process driven by energy supply and as such each term of the Penman-Monteith equation represents a separate heat source

for evaporation. Furthermore, the approach readily reveals some useful interpretations of the mathematical/physical attributes of key parameters of the Penman-Monteith equation. Although Chapter 2 essentially presents a review of the approach proposed by Monteith, the derivation places emphasis on basic assumptions and attempts to fill gaps (mainly in terms of providing extended discussion) in places where the original discussion was concise.

The derivation presented in Chapter 2 produces a small system of equations for computing vapor and heat flux from a source/sink surface to the ambient air and the change in air temperature during evaporation. However, another important physical process integral to evaporation and is inextricably coupled to the vapor and heat transfer processes is that of momentum transfer. Natural evaporation is generally conceptualized as a process driven mainly by near surface turbulent transfer and transport processes, caused by wind induced forced convection. Thus, the momentum transfer equation is key to developing expressions for the resistance parameters of vapor and heat transfer as well (which are parameters of the system of equations derived in Chapter 2). Accordingly, the derivation of the equation for the resistance to momentum transfer (or the aerodynamic resistance parameter) and its subsequent adaptation for use in evaluating the resistance parameters to vapor and heat transfer, under a suitably defined atmospheric condition, is described in Chapter 3.

Remarkably, the latent heat flux, sensible heat flux, final air temperature, and Δ equations derived in Chapter 2 cannot be evaluated directly, because the value of Δ is not known. Thus, numerical solutions are presented and evaluated in Chapter 4. Taking advantage of the mathematical structure of the system of equations, alternative numerical solutions with varying

degrees of complexity (consisting of four distinct algorithms, referred here as model 1, 2, 3, and 4) were developed and evaluated in Chapter 4. Results of model verification showed that each of the alternative models produced outputs practically identical and also in close agreement with a reference solution. Furthermore, intercomparison of the alternative models based on the criteria of numerical efficiency and robustness suggests that each model represents a comparable alternative, to any of the other models, for estimating evaporation. However, owing to its simplicity, model 1 is recommended here for further use.

A comparison of the outputs of model 1 with those of the conventional model (i.e., the approach widely used to evaluate the Penman-Monteith and related equations) shows that differences in the approaches of the two models has maximum effect on sensible heat flux estimates (where the mean absolute residual is 18.1%), a negligible effect on estimates of the final air temperature (with an average residual of 0.7%), and a limited effect on latent heat flux estimates, in which the mean residual is 8.2%.

Because both model 1 and the conventional model involve a level of approximation in the determination of Δ , a direct comparison of the two models cannot provided answer to the question: which model is more accurate? Thus, the current study does not try to answer that question. However, considering the latent heat flux, ℓ_f (which is the practically more important of the evaporation variables), the relatively small average percent residual (of 8.2%) suggests that the margins of error of the ℓ_f estimates obtained with model 1 and the conventional model may not be significantly different. This suggests that, at least for a good set of applications, the use of both the conventional model and model 1 can be considered equally valid from the standpoint of accuracy. However, the conventional model is preferred here, because of its simplicity.

Nonetheless, it ought to be emphasized that the study presented here is based on a limited data and hence a conclusive deduction needs to await a more comprehensive follow up study.

Chapter 2. Derivation of the Penman-Monteith system of equations with the thermodynamic approach: A review and theoretical development

2.1. Introduction

Evaporation from a cropped field is a complex physical process controlled by the interplay of weather, crop, and soil factors (Penman, 1948). A comprehensive physically based modeling of evaporation from a soil-canopy system requires a numerical solution of the equations describing the coupled processes of soil-water dynamics and the transfer and transport of vapor, heat, and momentum through the soil-crop-atmosphere continuum, taking into account wind induced advection and forced and free convection. Such a model is not only complex and demanding in terms of the type, resolution, and volume of input data, but also the resources required for acquiring and maintaining one for use in routine estimation of evaporation could be prohibitive.

Typically, for agricultural water management purposes, steady-state evaporation is computed with explicit equations as a function of, time averaged, measurable weather data. Equations of evaporation are developed based on varying degrees of approximations regarding the dominant factors and mechanisms that derive the vapor and heat transfer processes from the soil-canopy complex to the atmosphere. For instance, the Bowen ratio method, a method based on surface energy balance principles, is used to estimate evaporation from measurements of net solar radiation, soil heat flux, and gradients of air temperature and vapor pressure (Brutsaert, 1982; Monteith and Unsworth, 2013; Jensen and Allen, 2016). The aerodynamic method (e.g.,

Thornthwaite and Holzman, 1942) defined in terms of vapor pressure and wind speed is another approach for evaporation estimation from measured weather data. Additional empirical approximations that rely on limited sets of weather data for use in areas where a complete set is unavailable exist (Doorenbos and Pruitt, 1977).

An equation that combines the energy balance and aerodynamic components was derived by Penman (1948). Monteith (1965) extended the Penman equation into the form widely known as the Penman-Monteith equation, which has the same general form as its precursor, but involves partitioning of the leaf-boundary-layer resistance to vapor transfer into: (i) stomatal resistance for an isolated single leaf or canopy resistance for a uniform stand of vegetation and (ii) resistance to the transfer of vapor across the turbulent boundary layer. Note that partitioning of the resistance to vapor transfer along this line was also described by Penman (1953). The Penman-Monteith equation is generally considered as the most comprehensive steady-state evaporation equation and as such it was recommended as the standard method by FAO (Allen et al., 1998) and ASCE (Jensen and Allen, 2016).

A quick survey of the literature suggests that there are two approaches to the derivation of the Penman-Monteith equation. The most widely used approach (e.g., Jensen and Allen, 2016), broadly patterned after that of Penman (1948), generally considers evaporation as a vapor, heat, and momentum transfer process between a source/sink surface and a point (at measurement height) in the ambient air current. The rate (of vapor or heat) transfer is directly proportional to the respective potential differences and inversely related to the resistances across the points. During evaporation, the temperature of the exchange surface is considered constant. While the interface between the surface and the ambient air is presumed saturated, the air is generally considered unsaturated, although this may not necessarily be a requirement. Furthermore, the

measurement height ought to be sufficiently small for the air temperature not to be significantly different from the surface temperature.

An alternative approach proposed by Monteith (1965, 1981), on the other hand, conceptualizes evaporation as a physical process that produces changes in the thermodynamic state of the ambient air. A formulation that allows evaporation to be described in terms of (or from the vantage point of) the changes it introduces into the energy states of a thermodynamic system. Accordingly, a stationary air parcel that exchanges vapor and heat only with a source/sink surface is considered here to constitute a thermodynamic system. The approach proposes a framework in which evaporation is defined in terms of a pair of formal thermodynamic subprocesses (consisting of adiabatic cooling and diabatic heating) that leads to changes in the (energy) state of the air parcel in ways that are readily quantifiable. The derivation proceeds in two steps. As an initial approximation, first a form of the Penman-Monteith equation applicable to a scenario involving the interaction between a wet surface and a stationary ambient air is derived. The resultant equation is then modified, in a subsequent step, through the introduction of transfer (resistance) parameters to account for the effects of, wind-surface interactions and crop-physiological response to atmospheric conditions, on natural evaporation from a cropped field (where evaporation occurs at a potential rate). Overall, the conventional approach to the derivation of the Penman-Monteith equation (e.g., Jensen and Allen, 2016) has the advantage of being mathematically straight forward and compact. The thermodynamic approach (Monteith, 1981), on the other hand, is revealing of key assumptions and concepts that are generally implicit in the conventional approach.

This chapter derives the Penman-Monteith equation following the thermodynamic conceptualization of the evaporation process (Monteith, 1981). While essentially reviewing

Monteith's approach, the development here lists a specific set of assumptions that the derivation is based on, attempts to fill gaps (mainly in terms of providing extended discussion in places where the original discussion appear succinct), and places emphasis on interpretations of the mathematical/physical attributes of the terms and parameters of the Penman-Monteith equation that the derivation reveals.

Important observations stemming from the thermodynamic based derivation of the Penman-Monteith equation presented here are: (i) The approach accentuates the notion that the Penman-Monteith equation is fundamentally a description of the process of vapor and heat transfer between a wet source/sink surface and a stationary ambient air and that the convective transfer of these physical quantities (due to wind-surface interactions) and crop-physiological response to atmospheric conditions, which occur during natural evaporation, are taken into account in an approximate sense through the introduction of transfer parameters. (ii) It shows that evaporation is a process essentially driven by energy (heat) supply and as such each term of the Penman-Monteith equation represents a separate heat source for evaporation, consisting of external heat flux for the diabatic process and sensible heat extant in the air at the start of evaporation for the adiabatic process. (iii) The use of conceptual charts, depicting the energy states of a suitably defined thermodynamic system, in the derivation reveals rather readily the mathematical/physical attributes of key parameters of the Penman-monteith equation. (iv) A close look at the charts, interestingly shows, that evaporation can occur with a surface temperature that is less than the air temperature, provided the surface temperature is greater than the dewpoint temperature. And (v) Description of the terms of the Penman-Monteith equation as adiabatic and diabatic components, arising from the thermodynamic formulation, appears to be a more precise interpretation of the physical meaning of the terms.

The resultant equations for evaluating latent heat flux, sensible heat flux, and the final air temperature, referred here as the Penman-Monteith system of equations, cannot be evaluated directly. Thus, numerical solutions to this system of equations is presented and evaluated in Chapter 4.

2.2. Evaporation as a thermodynamic process and assumptions

Unlike the conventional approach, where evaporation is treated as vapor and heat transfer process between two points in space (i.e., a point in the exchange surface and another one in the air current), the thermodynamic conceptualization of evaporation introduces a perspective whereby the process can be studied from the vantage point of the changes it induces in the thermodynamic properties (specifically, the latent heat and sensible heat contents) of the ambient air. Following Monteith (1965, 1981), natural evaporation is, thus, described here in terms of a pair of formal thermodynamic subprocess (namely, adiabatic cooling and diabatic heating) that leads to changes in the energy states of a suitably defined thermodynamic system in ways that are readily quantifiable. The system consists of a stationary air parcel, of specified vapor pressure and temperature, that is in contact with a source/sink surface and exchanges vapor and heat with it, but not with the ambient air.

The Penman-Monteith equation is then derived in two steps. First, a form of the equation that models evaporation into a stationary ambient air from a wet surface is derived based on analysis of the changes in the thermodynamic states of the air parcel using equations of state. In a subsequent step, transfer (resistance) parameters are introduced into the basic equations, accounting for the dynamic effects of wind-surface interaction (the surface here being an

effective virtual surface within the crop canopy that serves as the sink for momentum and sink/source for vapor and heat) and those of crop-physiological response to atmospheric conditions, on evaporation, leading to the Penman-Monteith equation.

The following set of assumptions are introduced here as the basis to the derivation of the Penman-Monteith equation with the thermodynamic approach. *(i)* The thermodynamic system that forms the basis to the derivation of the Penman-Monteith equation consists of an air parcel, of given mass and specified vapor pressure and temperature, that neither mixes nor exchanges heat and vapor with the surrounding air, but exchanges vapor and heat with a source/sink surface that it is in contact with. Analyses based on air parcels of indeterminate form and shape are widely used in thermodynamic studies of the atmosphere in meteorology (e.g., Monteith, 1981; Arya, 1991). The presumption here is that observations deduced on the evaporation process involving a suitably defined system, based on analysis of the changes in the thermodynamic states of the system, can be generalized for the ambient air spanning the entire evaporative surface. *(ii)* The analysis considers an idealized (virtual) surface as the effective sink for momentum and source/sink for vapor and heat and such a surface is termed here as the exchange surface, or simply as, the surface. And *(iii)* The exchange surface has no mass and heat capacity (e.g., Arya, 1991) and hence it cannot store thermal energy, instead energy incident on the surface in the form of solar radiation is reflected, radiated back, used to heat the soil, and used to heat the air parcel (a fraction referred here as the external heat flux into the air parcel). It is assumed here that the surface temperature and vapor pressure are constant during evaporation and each of the surface energy balance components represent a steady flux during an evaporation process. Furthermore, it is presumed that there exist no temperature, vapor pressure, and energy flux gradient across the surface.

2.3. Evaporation into a stationary air parcel

2.3.1. Description of concepts

In order to develop the equations for evaporation into a stationary ambient air, consider a scenario in which water evaporates, from a wet source/sink surface, into a thermodynamic system consisting of a stationary air parcel of given mass, vapor pressure, and temperature. The thermodynamic state of such a system can be specified in terms of its latent and sensible heat contents, which in turn can be represented in terms of surrogate variables: vapor pressure, e , and temperature, T , respectively (Monteith, 1965). It, thus, follows that a vapor pressure versus temperature chart, akin to a simplified conceptual psychrometric chart, (sketched in Figure 1) can be used to help describe the evaporation process into a stationary air parcel. It ought to be emphasized here that the use of elements of the psychrometric chart and related terms in the, following, description of evaporation is only conceptual.

Figure 1 depicts some useful thermodynamic states of the air and pertinent processes superimposed on the simplified conceptual psychrometric chart. Two key features of the chart are the wet-bulb temperature line, $w(T)$ (corresponding to the initial thermodynamic state of the air, which is labeled as α , (T_0, e_0)), and the saturation vapor pressure curve, $e_s(T)$. As can be noted from Figure 1, the wet-bulb temperature line contains the states α and β , $(T_w, e_s(T_w))$, and the negative of its slope, γ (KPa/ $^{\circ}$ K), is the psychrometer constant. The saturation vapor pressure curve, on the other hand, contains the points ς , β , χ , σ , and ζ . Points ς , β , χ , and ζ define states of saturation, of the air parcel, at the dewpoint temperature, T_d , wet-bulb temperature, T_w , the measured air temperature, T_0 , and the final air temperature, T_a , respectively. Furthermore, the

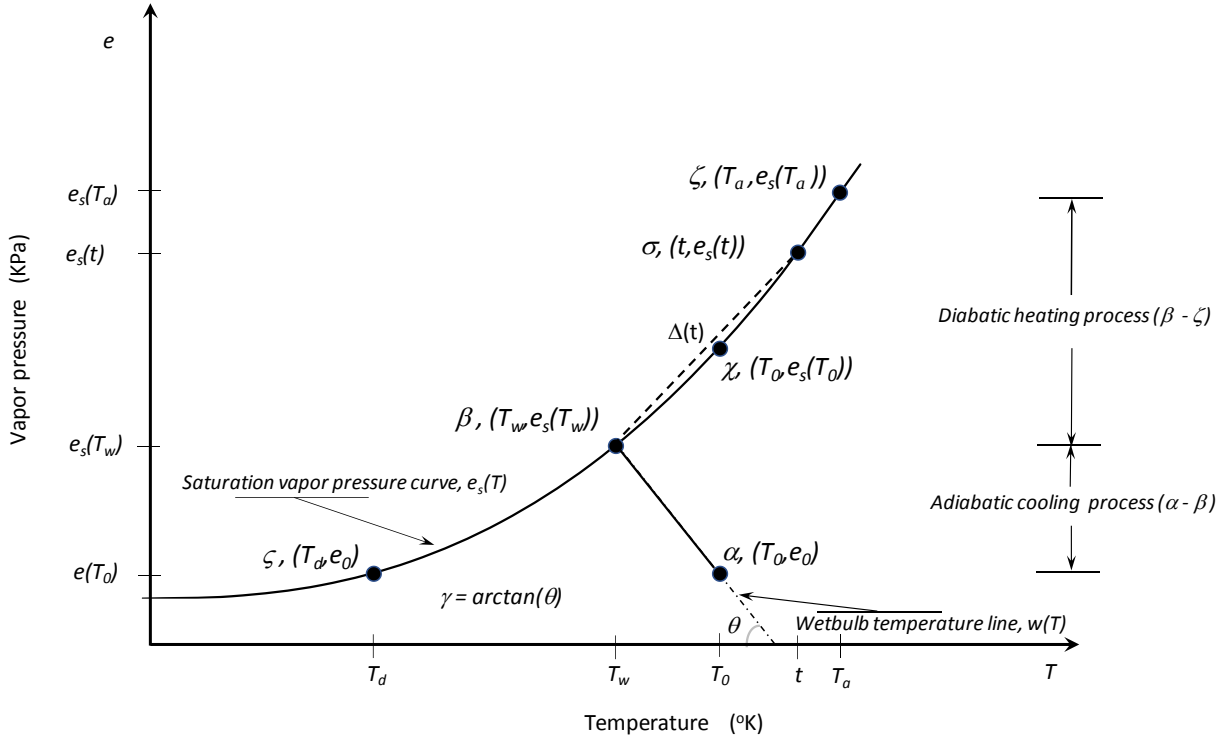


Fig 1. A simplified conceptual psychrometric chart depicting adiabatic cooling of the air parcel (from state α to β) and diatomic heating of the air in a state of saturation (from β to ζ). Note: T (°K) is temperature, T_d (°K) is the dewpoint temperature, T_w (°K) is the wet-bulb temperature, T_0 (°K) is the measured air temperature, t (°K) defines variable air temperature in the range $T_w \leq t \leq T_a$, e_s (KPa) is saturated vapor pressure at a specified temperature, e_0 or $e(T_0)$ (KPa) is measured vapor pressure, and γ is the psychrometer constant (KPa/°K)

point σ , $(t, e_s(t))$, is a state of saturation at some temperature t , where $T_w \leq t \leq T_a$, and $\Delta(t)$ is the slope of the line connecting state β with that of σ , which is a key parameter in simplifying the resultant equation. Although in Figure 1 the final air temperature, T_a , is greater than the measured dry-bulb temperature, T_0 , this, nonetheless, is not a requirement. Note that points α , (T_0, e_0) , and ζ , $(T_a, e_s(T_a))$, are the initial and final thermodynamic states, respectively, of the air parcel.

The change in the thermodynamic state of the air from α to that of ζ involves change in the total heat (i.e., the sum of the sensible and latent heat) content of the air parcel. Energy

differential between two states of a thermodynamic system is independent of the path taken and depends only on the initial and final states of the system defined by the thermodynamic properties (e.g., Rajput, 2007). Based on this principle, Monteith (1965) proposed a conceptual framework that describes the physics underlying natural evaporation, from a wet surface into a moist air parcel of specified vapor pressure and temperature, in terms of a pair of formal thermodynamic subprocesses (Figure 1), consisting of adiabatic cooling of the air driven by sensible heat extant in the air at the start of evaporation and diabatic heating in a state of saturation caused by external heat flux from the surface. Note that implicit in this description is the assumption that conversion of heat into work done by the air parcel is considered negligible, possibly on account of the consideration that evaporation takes place under constant pressure and the variation in air temperature (during evaporation) is sufficiently small for the density of the air parcel to be adequately represented by an average value.

Adiabatic cooling of a moist, yet unsaturated air parcel, is depicted in Figure 1 by the path connecting states α and β along the wet-bulb temperature line, $w(T)$. It can be observed from Figure 1 that the sensible heat extant in the air parcel at the start of evaporation (state α) is steadily reduced, resulting in a corresponding decrease in the air temperature from that of the measured temperature, T_0 , towards the wet-bulb temperature, T_w . However, heat does not cross the boundary of the air parcel (adiabatic process), instead it is used to evaporate water from the wet boundary of the parcel, resulting in an increase in the latent heat content of the air parcel and hence its vapor pressure from $e(T_0)$ towards $e_s(T_w)$. It thus follows, in the adiabatic process, the reduction in the sensible heat content of the air parcel is exactly balanced by the gain in latent heat. The processes of adiabatic cooling and humidifying of the air parcel continue eventually

leading to saturation at the wet-bulb temperature, $e_s(T_w)$, which is represented by the thermodynamic state, β .

By contrast, the diabatic heating process is represented by the path connecting states β and ζ along the saturation vapor pressure curve, $e_s(T)$, and involves both vapor and heat transfer between the exchange surface and the air parcel. For the scenario considered in Figure 1, where $T_a > T_w$, the external heat flux into the air parcel provides the energy that leads to heating of the air in a state of saturation. In other words, under the scenario in which T_a exceeds T_w , both the sensible heat and latent heat contents of the air parcel and its temperature and vapor pressure would increase.

If, on the other hand, the final air temperature is less than the wet-bulb temperature, i.e., $T_w > T_a$, it can then be observed from Figure 1 that diabatic cooling would occur. In other words, there would occur an outflow of both sensible and latent heat from the air parcel, leading to a decrease in the sensible and latent heat contents of the air parcel. However, for simplicity, the derivation presented here will be based on the assumption that $T_a > T_w$. It can be shown that the form of the equation obtained as such is directly applicable to the scenario where $T_w > T_a$, the only requirement is that the external heat flux needs to have a negative algebraic sign, indicating that the air parcel is losing heat to the exchange surface. Notably, a close look at the simplified conceptual psychrometric chart reveals that evaporation can occur even when the final air temperature, T_a , is less than the measured air temperature, T_0 , provided T_a is greater than the dewpoint temperature. It will be shown later that the final (equilibrium) air temperature is equal to the surface temperature.

Note that evaporation is considered here as a steady-state process. It is, therefore, important to observe that the adiabatic cooling and diabatic heating processes should not be

perceived as consecutive thermodynamic phases in time. They are only used in the conceptual definition of evaporation in terms of the constituent thermodynamic subprocesses, the identification of corresponding sources of heat, and specification of the changes in the states of the system in ways that are readily quantifiable. The preceding formulation of natural evaporation as a thermodynamic process (Monteith, 1981) will now be used in subsequent derivation of the Penman-Monteith equation.

2.3.2. Equations

The preceding discussion summarizes the thermodynamic formulation of the evaporation process using a relatively simple, and more familiar, conceptual chart that defines the state of the system in terms of vapor pressure, e , and temperature, T . However, Monteith (1981) used a conceptual two-dimensional chart (that specifies the thermodynamic states of an air parcel in terms of its latent heat and sensible heat contents per unit mass) to derive a form of the Penman-Monteith equation that models evaporation into a stationary ambient air (of specified e_0 and T) from a wet exchange surface. Here, on the other hand, the analysis is based on the latent heat, $q(T)$, and sensible heat, $\ell(T)$, contents of a unit volume of the air parcel, Figure 2. This is predicated on the consideration that evaporation takes place under constant pressure and the range of temperature changes during evaporation (i.e., the interval between T_w and T_a) are sufficiently small for volume variations to be considered not significant and hence for the air density to be represented adequately in terms of an average value.

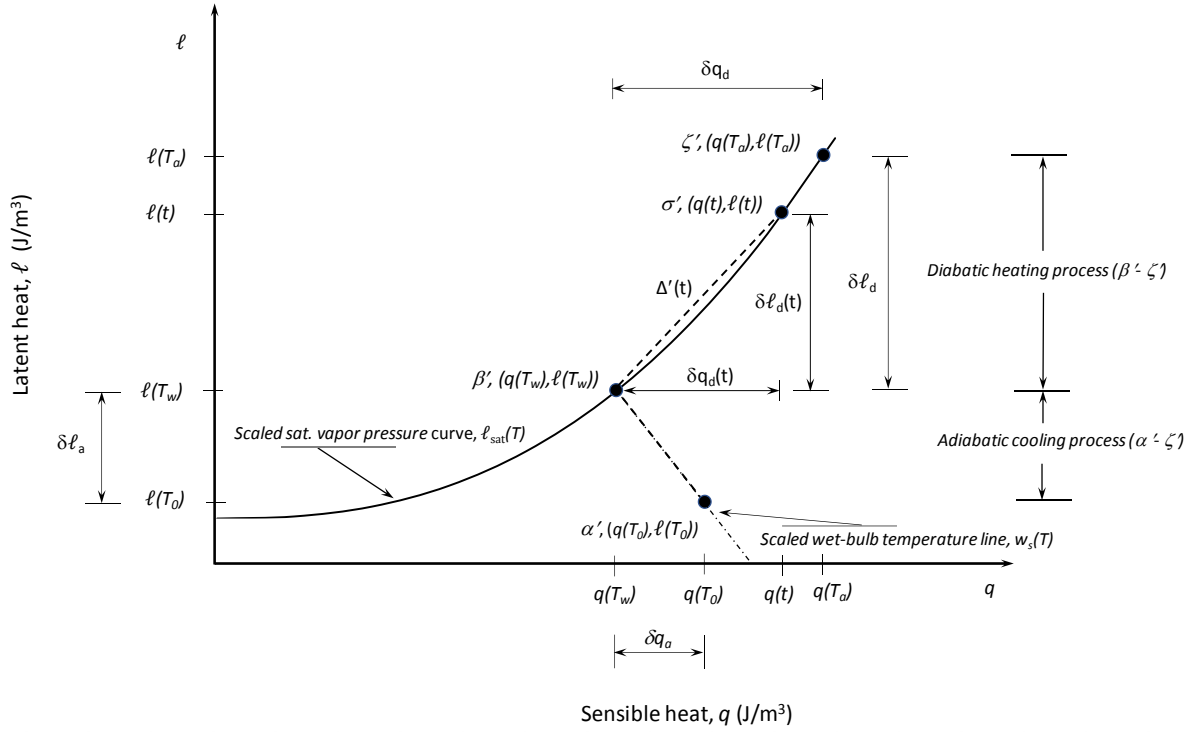


Fig 2. Simplified chart depicting the thermodynamic states of an air parcel in an evaporation process (sensible heat and latent heat content space, adapted from Monteith, 1981)

A close look at Figure 2, in light of the unique dependence of ℓ and q on temperature, T , along the w_s line and the ℓ_{sat} curve, reveals that Figure 2 represents a scaled version of the conceptual psychrometric chart, depicted in Figure 1, where the scaling constants are ρc_p for the abscissa and $\lambda \rho \epsilon / p$ for the ordinate. It can, thus, be observed that there is a one-to-one mapping between the thermodynamic states in Figure 1 and those of Figure 2. Accordingly, it can be observed that states α' , β' , σ' , and ζ' in Figure 2 correspond to states α , β , σ , and ζ in Figure 1, respectively. The adiabatic cooling process is represented by the path connecting states α' and β' along the w_s line and diabatic heating of the air parcel, while maintaining saturation, is represented by the path connecting β' and ζ' along the ℓ_{sat} curve. Furthermore, the $\Delta'(t)$

parameter represents the slope of the line connecting states β' and σ' at some temperature t , where $T_w \leq t \leq T_a$.

The thermodynamic equation of state for latent heat, $\ell(T)$, can now be defined along the scaled wet-bulb temperature line, w_s , and the scaled saturation vapor pressure curve, ℓ_{sat} , as

$$\ell(T) = \lambda \rho \xi(T), \quad \text{where } \xi = \frac{\epsilon}{(p - e) + \epsilon e} e(T) \approx \left(\frac{\epsilon}{p}\right) e(T) \quad (1)$$

Similarly, the equation of state for sensible heat, $q(T)$, can be given as

$$q(T) = \rho c_p T \quad (2)$$

where $\ell(T)$, J/m³, and $q(T)$, J/m³, are latent heat and sensible heat contents per unit volume of the air parcel, respectively; ρ (Kg/m³) is the density of air; λ (J/Kg) is the latent heat of vaporization of water; ξ (Kg-vapor/Kg-moist-air) is the specific humidity of the air; ϵ (-) is the ratio of the molar mass of vapor to that of dry air and is set to 0.622; p (KPa) is air pressure; e (KPa) is vapor pressure; c_p (J/Kg-°K) is the specific heat of air at constant pressure; and T (°K) is temperature.

The change in the latent heat content of a unit volume of the air parcel, $\delta\ell$ (J/m³), during evaporation, can be expressed as the sum of the adiabatic, $\delta\ell_a$, and diabatic, $\delta\ell_d$, components

$$\delta\ell = \delta\ell_a + \delta\ell_d \quad (3)$$

Similarly, the corresponding change in the sensible heat content of a unit volume of the air parcel, δq (J/m³), can be given as the sum of the adiabatic, δq_a , and diabatic, δq_d , fractions

$$\delta q = \delta q_a + \delta q_d \quad (4)$$

Expressions for the changes in the latent heat, $\delta \ell$, and sensible heat, δq , contents of a unit volume of the air parcel as a function of readily measurable weather parameters will now be derived.

2.3.2.a. Evaporation into a moist but unsaturated air, adiabatic cooling process

Consider an air parcel that is cooled from the measured air temperature, T_0 , to its wet-bulb temperature, T_w , adiabatically, while its vapor pressure is increased from e_0 (or $e(T_0)$) to $e_s(T_w)$, which is represented by the path between α' and β' in Figure 2. As noted earlier, in the adiabatic process, the changes in the latent heat and sensible heat contents of a unit volume of the air parcel ($\delta \ell_a$ and δq_a , respectively) are equal in magnitude, but are of opposite algebraic sign. Thus, the heat balance of the air parcel, for the adiabatic process, can be given as

$$\delta \ell_a + \delta q_a = 0 \quad (5)$$

The decrement in the sensible heat content per unit volume of the air parcel, δq_a , defined along the path α' to β' , Figure 2, can be expressed as

$$\delta q_a = \rho c_p (T_w - T_0) \quad (6)$$

The corresponding increment in the latent heat content per unit volume of the air parcel, $\delta \ell_a$, can be given as

$$\delta\ell_a = \frac{\lambda\rho\epsilon}{p}(e_s(T_w) - e_0) \quad (7)$$

Equations 6 and 7 represent potential differences between the initial and final thermodynamic states defining the adiabatic cooling process. Noting that the dry-bulb temperature can be measured more readily and accurately than the wet-bulb temperature, an equation that relates the change in the latent heat content of the air parcel in the adiabatic process, $\delta\ell_a$, with the vapor pressure deficit at the measured temperature, T_0 , will now be presented.

Noting that $\delta\ell_a = -\delta q_a$ (from Eq. 5) and equating the negative of the right-hand side of Eq. 6 with that of Eq. 7 and simplifying the resultant expression yields

$$e_s(T_w) - e_0 = \gamma(T_0 - T_w), \quad \text{where } \gamma = \frac{p c_p}{\lambda \epsilon} \quad (8)$$

Another key parameter (i.e., in addition to γ), which is labeled in Figure 2 as $\Delta'(t)$ and represents the slope of a linear equation that relates point $\beta'(q(T_w), \ell(T_w))$ on the $\ell_{sat}(T)$ curve with any other point, $\sigma'(q(t), \ell(t))$, on the same curve will now be introduced

$$\ell_{sat}(T_w) = \ell_{sat}(t) - \Delta'(t)(q(t) - q(T_w)) \quad (9)$$

where t is a temperature that varies in the range $T_w \leq t \leq T_a$. Substituting $\lambda\rho\epsilon e_s(t)/p$ for $\ell_{sat}(t)$ and $\rho c_p t$ for $q(t)$ into Eq. 9 and simplifying yields

$$e_s(T_w) = e_s(t) - \Delta'(t)\gamma(t - T_w) \quad (10)$$

Based on the definition of the slope of a line, it can be shown that $\Delta'(t)$ is equal to $\Delta(t)$ (Figure 1) scaled by the reciprocal of the psychrometer constant

$$\Delta'(t) = \frac{\Delta(t)}{\gamma} \quad (11)$$

Combining Eqs. 10 and 11 and simplifying yields

$$e_s(T_w) = e_s(t) - \Delta(t)(t - T_w) \quad (12)$$

Furthermore, substituting the right-hand side of Eq. 12 into Eq. 8 and rearranging terms yields

$$e_s(t) - e_0 = \Delta(t)(t - T_w) + \gamma(T_0 - T_w) \quad (13)$$

Setting $t = T_0$ and rearranging results in

$$T_0 - T_w = \frac{e_s(T_0) - e_0}{\Delta(T_0) + \gamma} \quad (14)$$

Note that Eq. 14 is an exact expression, provided $\Delta(T_0)$ is evaluated at an appropriate temperature, τ , such that

$$\Delta(T_0) = \left. \frac{\partial e_s(T)}{\partial T} \right|_{T=\tau}, \text{ where } T_w \leq \tau \leq T_0 \quad (15)$$

Combining Eqs. 14 and 6 and then substituting Δ_a for $\Delta(T_0)$ in the resultant expression (while noting that in the adiabatic cooling process, $\delta\ell_a = -\delta q_a$), yields the expression for $\delta\ell_a$ in terms of the measured air temperature

$$\delta\ell_a = \rho c_p \frac{e_s(T_0) - e_0}{\Delta_a + \gamma} \quad (16)$$

The right-hand side of Eq. 16 is the adiabatic term of a form of the Penman-Monteith equation that models evaporation into a stationary ambient air from a wet surface. Remarkably, it can be shown that Eq. 16 is of the same form as the expression for latent heat associated with an isolated system cooled adiabatically to its thermodynamic wet-bulb temperature (e.g., Monteith and Unsworth, 2013).

2.3.2.b. Evaporation into saturated air, diabatic heating in a state of saturation

In contrast to the adiabatic cooling process, in the diabatic heating process heat transfer between the exchange surface and the air parcel is permissible. As noted earlier, in the scenario considered here (where $T_a > T_w$), diabatic heating (driven by a steady external heat flux into the air) leads to an increase in both the sensible and latent heat contents of the air parcel and hence to a change in the thermodynamic state of the air parcel from that of $\beta'(T_w, e_s(T_w))$ to $\zeta'(T_a, e_s(T_a))$ along the scaled saturation vapor pressure curve, Figure 2. It, thus, follows that the answer to the question of finding an expression for the change in the latent heat content of the air (along the path β' to ζ') boils down to the determination of the proportionality factor with which the external heat flux is partitioned into latent and sensible heat contents.

It follows from the preceding discussion that the heat balance of the air parcel in the diabatic process can be given as

$$Q = \delta\ell_d + \delta q_d \quad (17)$$

where Q is the external heat input into a unit volume of the air parcel that leads to a change in the total heat content of the air parcel (J/m^3). As can be noted from Figure 2, $\delta\ell_d$ (J/m^3) and δq_d

(J/m³) are, respectively, increments in the latent heat and sensible heat contents of a unit volume of the air parcel as it is heated in a state of saturation.

Considering a linear equation (of the form given in Eq. 9) that relates the point $\beta'(T_w, e_s(T_w))$ on the scaled saturation vapor pressure curve with that of $\zeta'(T_a, e_s(T_a))$, it can be observed (from Eq. 11) that for $t = T_a$, $\Delta'(T_a) = \Delta(T_a)/\gamma$. Substituting Δ_d for $\Delta(T_a)$ and noting that $\delta q_d = Q - \delta \ell_d$ (Eq. 17), it follows from Figure 2 that

$$\frac{\Delta_d}{\gamma} = \frac{\delta \ell_d}{\delta q_d} = \frac{\delta \ell_d}{Q - \delta \ell_d} \quad (18)$$

Simplifying, yields

$$\delta \ell_d = \frac{\Delta_d Q}{\Delta_d + \gamma} \quad (19)$$

2.3.2.c. Combined equations: A form of the Penman-Monteith equation that models evaporation into a stationary ambient air from a wet surface and related equations

The linear equations defined in terms of Δ_a (e.g., 12) and Δ_d (the corresponding equation is not shown here for brevity) are exact expressions, provided they are computed subject to the requirement stated in Eq. 15. However, in the interest of keeping the resultant equation for $\delta \ell$ simpler, it is assumed here that an effective average Δ , somewhere in the interval $[\Delta_a, \Delta_d]$, can be used in place of both Δ_a and Δ_d . Note that this is predicated on the assumption that the temperature interval, $T_a - T_0$, during a natural evaporation process is sufficiently small, such that Δ_a is close enough to Δ_d , and hence a constant, Δ , can be used in place of Δ_a and Δ_d without compromising accuracy.

Now, substituting Δ for Δ_a and Δ_d in Eqs. 16 and 19, respectively, and combining the resultant expressions with the equation for $\delta\ell$ (Eq. 3) yields the change in the latent heat content of a unit volume of a stationary air parcel during an evaporation process from a wet surface

$$\delta\ell = \lambda E = \frac{\Delta Q}{\Delta + \gamma} + \rho c_p \frac{e_s(T_0) - e_0}{\Delta + \gamma} \quad (20)$$

In Eq. 20, E is the cumulative evaporation into a unit volume of the ambient air (Kg/m^3). Note that alternative methods for computing Δ are presented in Chapter 4.

Now, following the same approach as that used to deduce the equation for $\delta\ell_d$ (Eqs. 17 to 19), it can be shown that the expression for δq_d (J/m^3) can be given as

$$\delta q_d = \frac{\gamma Q}{\Delta + \gamma} \quad (21)$$

Substituting Eq. 21 and the negative of Eq. 16 into Eq. 4 yields the overall change in the sensible heat content of a unit volume of the air parcel, δq , as

$$\delta q = \frac{\gamma Q}{\Delta + \gamma} - \rho c_p \frac{e_s(T_0) - e_0}{\Delta + \gamma} \quad (22)$$

An equation for the final temperature of the air parcel, T_a , can be deduced by combining Eq. 22 with the equation of state for sensible heat (a form analogous to Eq. 6), then setting the lower and upper limits of the temperature range to T_0 and T_a , respectively, and solving the resultant expression for T_a , which yields

$$T_a = T_0 + \frac{\gamma Q}{\rho c_p (\Delta + \gamma)} - \frac{e_s(T_0) - e_0}{\Delta + \gamma} \quad (23)$$

In the development of Eqs. 20, 22, and 23, evaporation is treated as a process of vapor and heat transfer from a wet exchange surface into a stationary ambient air. However, natural evaporation from a cropped field, where water availability is not a limiting factor, occurs under a more complex set of conditions. Typically, there is advective transport due to wind; interaction between wind and the soil-canopy system which leads to the convective transfer and transport of momentum, heat, and vapor by turbulent eddies; and crop-physiological response to atmospheric conditions. The Penman-Monteith equation will now be developed accounting for these effects, although in an approximate sense.

2.4. The Penman-Monteith equation

2.4.1. Evaporation described in terms of sensible heat and latent heat flux space and equations

In subsequent development, we will continue to make use of Monteith's formulation that frames evaporation as a thermodynamic process that can be described by tracking the changes in the state of a stationary air parcel, of given mass and vapor pressure and temperature, in contact with an exchange surface. The assumptions made earlier regarding the modes of interaction of the air parcel with the exchange surface and the surrounding air remains in place. In addition, in subsequent discussion, the consideration, indicated earlier, that natural evaporation occurs under constant air pressure and that the range of variation of temperature is limited (hence the variation in the density of the air is sufficiently small to be represented by an average value) holds here as well.

Furthermore, the following simplifying assumption are introduced here as the basis to the development of the Penman-Monteith equation accounting for the dynamic effects of wind and soil-canopy system interaction effects on natural evaporation: (i) A cropped field with a uniform stand and optimal vegetative growth and no limitations in the availability of water is considered, thus, evaporation occurs at the potential rate. (ii) The turbulent transfer of vapor, heat, and momentum between an exchange surface and the ambient air (produced mainly by wind-surface interaction effects) and crop-physiological response (to atmospheric conditions) are considered the principal coupled mechanisms that control evaporation. (iii) A virtual surface, situated within the crop canopy, serve as the effective sink for momentum and source/sink for vapor and heat. (iv) The convective transfer of heat and vapor between the source/sink surface and the air parcel and crop-physiological response to atmospheric conditions can be modelled sufficiently accurately by combining the equations for potential difference (defined in terms of the thermodynamic equations of state, Eqs. 1 and 2) with an appropriate set of transfer (resistance) parameters. And (v) The cropped field covers a sufficiently expansive area (with negligible edge effect) and hence the transfer of physical quantities of interest is one-dimensional in the vertical direction.

Evidently, the introduction of the transfer parameters into the basic equations implies that the interest here is not cumulative heat gains/losses by the air parcel during an evaporation process, as was the case in the preceding development. Instead, it is in heat fluxes into/from the air parcel in the diabatic process and of course in the rate of conversion of thermal energy (from sensible heat to latent heat) within the air parcel in the adiabatic process. Accordingly, following Monteith (1981), Figure 2 will now be mapped onto a two-dimensional space in which the abscissa is sensible heat flux, $q_f(T)$ (W/m^2), and the ordinate will be latent heat flux, $\ell_f(T)$

(W/m²), as shown in Figure 3. Note that Monteith (1981) did not specifically present Figure 3 in any form, but described its features and applied them to deduce the Penman-Monteith equation.

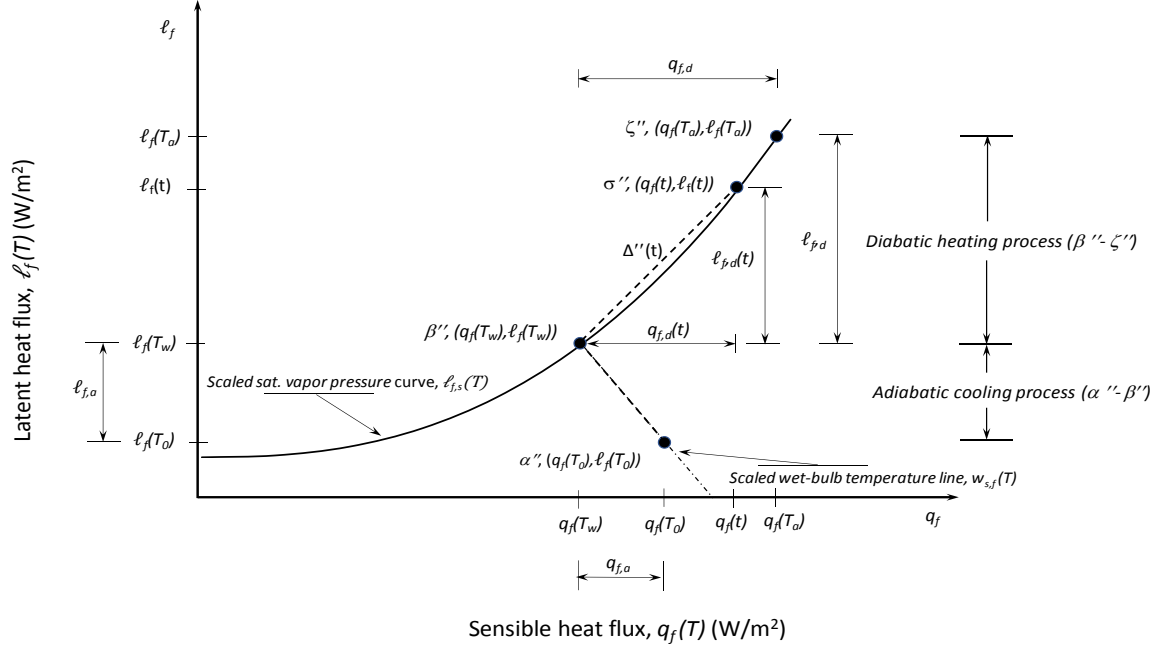


Fig 3. Simplified chart depicting the thermodynamic states of an air parcel in an evaporation process (sensible and latent heat flux space)

Along the scaled wet-bulb temperature line, $w_{s,f}$, and that of the scaled saturation vapor pressure curve, $\ell_{f,s}$, of Figure 3, the latent heat and sensible heat fluxes can be expressed as

$$q_f(T) = \frac{q(T)}{r_h} \quad \text{and} \quad \ell_f(T) = \frac{\ell(T)}{r_v} \quad (24)$$

In Eq. 24, $q(T)$, J/m³, and $\ell(T)$, J/m³, are latent and sensible heat contents of a unit volume of the air parcel, respectively, defined in terms of Eqs. 1 and 2, and r_v (s/m) and r_h (s/m) are transfer parameters. Further discussion on the transfer parameters and the equations used to evaluate them will be presented later.

2.4.2. Physical interpretation of the flux equations and the use of flux as a thermodynamic property

In principle, fluxes are defined across potential differences. However, that appears not to be the case in Eq. 24. A question that may, thus, arise here is: what physical meaning can be assigned to the fluxes defined in Eq. 24. The sensible heat flux at some temperature, T , $q_f(T)$ or $q(T)/r_v$, can be interpreted as the amount of heat that needs to flow into the air parcel, per unit area of the surface and in a unit time of interest, in order to raise the heat content of a unit volume of the air to such a level that its temperature is increased from a reference temperature (absolute zero, 0°K , considered ground state) to T , given the r_h value. Similarly, the difference between two levels of the sensible heat flux, say between $q_f(t)$ and $q_f(T_w)$ (which is labeled as $q_{f,d}(t)$ in Figure 3), represents the quantity of heat that must flow into the air parcel (across a unit area of the surface and in a unit time of interest) so as to raise the heat content of a unit volume of the air by such an amount that its temperature is increased from T_w to t , given the resistance parameter. Note that same observation can be made with regard to the latent heat flux defined in Eq. 24.

Another important point that may require a closer examination here is the use of fluxes as state variables, which appears unconventional. Given that fluxes are less of intrinsic properties of the air parcel, a question might be raised about the implications of using fluxes as state variables. Although Figure 3 is less intuitive than that of Figure 2, it can be shown that there is a one-to-one mapping between the set of points in Figures 3 and that of Figure 2 and as such Figure 3 can be considered as an equally valid representation of the thermodynamic states of the air parcel as that of Figure 2.

As noted earlier, it is the unique dependence of latent heat, $\ell(T)$, and sensible heat, $q(T)$, contents of the air on temperature (along the wet-bulb temperature and the saturation vapor

pressure curve) that forms the basis of the relationship between Figures 1 and 2. It will now be shown that this property remains unchanged following the transformation involving the multiplication of $\ell(T)$ and $q(T)$ by the reciprocals of the respective resistance parameters, Figure 3.

For practical purposes, the transfer parameters, r_v and r_h , can generally be considered independent of temperature, and hence presumed constant for a given crop and stage of development (assuming optimal conditions) and wind condition. This implies that the abscissa of Figure 3 is only a scaled version of that of Figure 2, the scaling constant being $1/r_h$. Hence, values on the horizontal axis of Figure 3 would simply be shifted rightwards or leftwards by a factor of $1/r_h$ with respect to the corresponding values of the abscissa of Figure 2, depending on whether $r_h < 1$ or $r_h > 1$, respectively. Similarly, the ordinate of Figure 3 is equal to the ordinate of Figure 2 scaled by a constant factor of $1/r_v$. It can, thus, be readily observed that there is a one-to-one mapping between the set of points constituting the two-dimensional space of Figure 2 and those of Figure 3. In other words, every point (or thermodynamic state) in Figure 2 has a counterpart (and hence can be uniquely and equally represented) in Figure 3 (for instance, states α'' , β'' , σ'' , and ζ'' in Figure 3 correspond to states α' , β' , σ' , and ζ' in Figure 2). Note that the same reasoning can be used to show that Figure 3 is simply a scaled version of Figure 1. Furthermore, it can be observed that the scaled saturation vapor pressure curve, $\ell_{f,s}(T)$, in Figure 3 has the same general behavior with respect to temperature (a monotonic increasing convex function of temperature) as $e_s(T)$, because it is the $e_s(T)$ curve of Figure 1 scaled by a pair of constants: $\rho c_p/r_h$ for the abscissa and $\lambda\rho\epsilon/(r_v p)$ for the ordinate. Finally, it can be shown that the slope of the scaled saturation vapor pressure curve, Figure 3, is equal to the slope of the

saturation vapor pressure curve, Figure 1, scaled by the reciprocal of the modified psychrometer constant (to be defined later).

An important point here is that in Eq. 24, potential difference is defined as the difference between the energy levels of the thermodynamic states of the air parcel. By contrast, in the conventional approach to the derivation of the Penman-Monteith equation, the potential difference represents difference in (thermal) energy levels between two points in space, i.e., a point on the surface and another one in the air current. Thus, this raises the question whether the physical meaning assigned to the parameters r_v and r_h in the conventional formulation (as the resistance to vapor and heat transfer, respectively) and the method used to evaluate them can apply to the thermodynamic formulation. This may, perhaps, need to be looked into in some detail. However, following Monteith (1981), it is assumed here that r_v and r_h (henceforth referred to as resistance parameters to vapor and heat transfer, respectively) have the same physical meaning as in the conventional approach and can be evaluated with the same method as those used in the conventional approach.

2.4.3. Equations

The latent heat flux, ℓ_f (W/m^2), and the sensible heat flux, q_f (W/m^2), into the air parcel during an evaporation process can be expressed as the sum of the respective adiabatic and diabatic components

$$\ell_f = \ell_{f,a} + \ell_{f,d} \quad \text{and} \quad q_f = q_{f,a} + q_{f,d} \quad (25)$$

In Eq. 25, $\ell_{f,a}$ and $\ell_{f,d}$ are the adiabatic and diabatic fractions, respectively, of the latent heat flux and $q_{f,a}$ and $q_{f,d}$ represent the components of the sensible heat flux associated with the adiabatic

and diabatic processes, respectively. The latent heat and sensible heat fluxes expressed in terms of readily measurable weather parameters (including wind speed) and crop factors will now be derived.

2.4.3.a. Evaporation into a moist but unsaturated air: Adiabatic cooling process accounting for the convective transfer of heat and vapor and crop physiological response

For the adiabatic process, the flux form of the heat balance equation of the air parcel can be given as

$$\delta \ell_{f,a} + \delta q_{f,a} = 0 \quad (26)$$

Based on Eq. 24, the sensible heat flux associated with the adiabatic process, $q_{f,a}$, can be expressed as the quotient of the potential difference between states α' and β' , Figure 2, and the resistance to heat transfer

$$q_{f,a} = \rho c_p \frac{T_w - T_0}{r_h} \quad (27)$$

The corresponding latent heat flux associated with the adiabatic process, $\ell_{f,a}$, can be expressed as

$$\ell_{f,a} = \rho c_p \frac{e_s(T_w) - e_0}{\gamma^* r_h}, \quad \text{where } \gamma^* = \gamma \frac{r_v}{r_h} \quad (28)$$

In Eq. 28, γ^* is the modified psychrometer constant (KPa/ $^{\circ}$ K) and ρ , c_p , $e_s(T_w)$, e_0 , r_v , r_h , and γ were defined earlier.

Given that evaporation is considered here as an adiabatic cooling process, heat is not crossing the boundaries of the parcel. Thus, $q_{f,a}$ in Eq. 27 ought to be viewed as the equivalent

rate at which heat needs to flow from the air parcel onto a unit area of the surface per unit time, if the sensible heat content of a unit volume of the air parcel is to be reduced from the level at state α' to that at β' (Figure 2) through heat efflux from the air parcel onto the surface. Note that similar interpretations can be made for the latent heat flux, Eq. 28, as well.

With the view of keeping the form of the resultant equation simple, an alternative expression that relates $\ell_{f,a}$ with the vapor pressure deficit at the measured dry-bulb temperature, T_0 , will now be derived. Noting that $\ell_{f,a} = -q_{f,a}$ (from Eq. 26) and equating the right hand-side of Eq. 28 with the negative of the right-hand side of Eq. 27 and simplifying yields

$$e_s(T_w) - e_0 = \gamma^*(T_0 - T_w) \quad (29)$$

A linear equation (of the form given in Eq. 9) that relates the latent heat and sensible heat fluxes at states β'' and σ'' (along the scaled saturation vapor pressure curve, $\ell_{f,s}$, Figure 3) will now be introduced

$$\ell_f(T_w) = \ell_f(t) - \Delta''(t) (q_f(t) - q_f(T_w)) \quad (30)$$

Substituting $\lambda \rho \in e_s(t)/(r_v p)$ for $\ell_f(t)$ and $\rho c_p t/r_h$ for $q_f(t)$ into Eq. 30 yields

$$e_s(T_w) = e_s(t) - \Delta''(t) \gamma^*(t - T_w) \quad (31)$$

Based on the definition of the slope of a line, it can be shown that $\Delta''(t)$ is equal to $\Delta(t)$ (Figure 1) scaled by the reciprocal of the psychrometer constant

$$\Delta''(t) = \frac{\Delta(t)}{\gamma^*} \quad (32)$$

Combining Eqs. 31 and 32 yields

$$e_s(T_w) = e_s(t) - \Delta(t)(t - T_w) \quad (33)$$

Furthermore, substituting the right-hand side of Eq. 33 into Eq. 29 and rearranging yields

$$e_s(t) - e_0 = \Delta(t)(t - T_w) + \gamma^*(T_0 - T_w) \quad (34)$$

Setting $t = T_0$ and rearranging, results in

$$T_0 - T_w = \frac{e_s(T_0) - e_0}{\Delta(T_0) + \gamma^*} \quad (35)$$

Note that Eq. 35 is an exact expression, provided $\Delta(T_0)$ is evaluated at an appropriate temperature, τ , such that

$$\Delta(T_0) = \left. \frac{\partial e_s(T)}{\partial T} \right|_{T=\tau}, \quad \text{where } T_w \leq \tau \leq T_0 \quad (36)$$

Combining Eqs. 35 and 27 and then substituting Δ_a for $\Delta(T_0)$ in the resultant expression (while noting that in the adiabatic cooling process, $\ell_{f,a} = -q_{f,a}$), it can be shown that the adiabatic fraction of the latent heat flux, $\ell_{f,a}$, can be expressed as

$$\ell_{f,a} = \rho c_p \frac{e_s(T_0) - e_0}{(\Delta_a + \gamma^*)r_h} \quad (37)$$

2.4.3.b. Evaporation into saturated air: Diabatic heating in a state saturation accounting for the convective transfer of heat and vapor and crop physiological response

For the diabatic component of the evaporation process, the flux form of the heat balance of the air parcel can be given as

$$Q_f = \ell_{f,d} + q_{f,d} \quad (38)$$

where Q_f is the steady external heat flux into the air parcel (W/m^2), which is equal to the net surface heat flux minus the ground heat flux.

Considering a linear equation (of the form given in Eq. 30) that relates the point $\beta''(T_w, e_s(T_w))$ on the scaled saturation vapor pressure curve with that of $\zeta''(T_a, e_s(T_a))$, i.e., a scenario in which $\Delta''(t) = \Delta''(T_a)$; it can be readily observed (from Eq. 32) that $\Delta''(T_a) = \Delta(T_a)/\gamma$. Using Δ_d in place of $\Delta(T_a)$, it follows from Figure 3 that

$$\frac{\Delta_d}{\gamma^*} = \frac{\ell_{f,d}}{q_{f,d}} \quad (39)$$

Substituting the expression $Q_f - \ell_{f,d}$ for $q_{f,d}$ (from Eq. 38) into Eq. 39 and rearranging yields the expression for the diabatic fraction of the latent heat flux, $\ell_{f,d}$

$$\ell_{f,d} = \frac{\Delta Q_f}{\Delta + \gamma^*} \quad (40)$$

2.4.3.c. The Penman-Monteith system of equations

Combining Eqs. 37 and 40 with Eq. 25 yields the Penman-Monteith equation

$$\ell_f = \lambda E_f = \frac{\Delta Q_f}{\Delta + \gamma^*} + \rho c_p \frac{e_s(T_0) - e_0}{(\Delta + \gamma^*)r_h} \quad (41)$$

where E_f is evaporation flux into the air parcel (Kg/s-m^2). Similarly, the expression for the diabatic component of the sensible heat flux, $q_{f,d}$, can be given as

$$q_{f,d} = \frac{\gamma^* Q_f}{\Delta + \gamma^*} \quad (42)$$

Substituting Eq. 42 and the negative of Eq. 37 into the expression for q_f (Eq. 25) yields the equation for the net sensible heat flux into the air parcel, q_f , during evaporation

$$q_f = \frac{\gamma^* Q_f}{\Delta + \gamma^*} - \rho c_p \frac{e_s(T_0) - e_0}{(\Delta + \gamma^*)r_h} \quad (43)$$

An equation for the final temperature of the air parcel, T_a , can be deduced by combining Eq. 43 with the expression for sensible heat (a form analogous to Eq. 27), then setting the lower and upper limits of the temperature range to T_0 and T_a , respectively, and solving the resultant expression for T_a , which yields

$$T_a = T_0 + \frac{\gamma^* r_h Q_f}{\rho c_p (\Delta + \gamma^*)} - \frac{e_s(T_0) - e_0}{\Delta + \gamma^*} \quad (44)$$

It can be observed that the equation for the final air temperature, Eq. 44, is identical to the expression for the temperature of the exchange surface (e.g., Monteith and Unsworth, 2013). Considering that the surface temperature and the external heat flux into the air parcel are both held steady through the evaporation process, it can be observed that the equivalence of the expressions for the final air temperature and the surface temperature points to the fact that the surface temperature is the same as the equilibrium air temperature.

2.4.3.d. Resistance parameters

As noted earlier, the resistance parameters to vapor and heat transfer (r_v and r_h) are introduced into Eq. 24 to take into account the dynamic effects of turbulence generated by wind-surface interactions, and the effects of crop-physiological response to atmospheric conditions, on evaporation from a cropped field. However, the pathway for vapor transfer from a plant leaf to the atmosphere is considered to be composed of two distinct mechanisms: vapor transfer from the leaf through the stomata and cuticle to the laminar sublayer on the leaf surface by diffusion (controlled mainly by crop-physiological response to atmospheric conditions) followed by a convective vapor transfer across the turbulent boundary layer (Monteith, 1965 and 1981). Thus, Monteith(1965) described the resistance to vapor transfer, r_v , as the sum total of stomatal and cuticular resistance, r_s , and resistance to vapor transfer in the turbulent boundary layer, r_b (note that partitioning of the resistance to vapor transfer along this line was also described by Penman, 1953). For cropped fields with uniform vegetation stand, r_s is referred to as canopy resistance (Monteith and Unsworth, 2013). In agricultural applications it is described as the bulk surface resistance (Allen et al., 1998) and is evaluated as function of stomatal resistance and crop leaf area index. Accordingly, the parameter γ^* , Eq. 28, can be given as

$$\gamma^* = \gamma \frac{r_b + r_s}{r_h} \quad (45)$$

As can be noted from Eq. 45, Eqs. 41, 43, and 44 are defined in terms three different resistance parameters, the bulk surface resistance to vapor transfer, r_s , the turbulent boundary layer resistance to vapor transfer, r_b , and the resistance to heat transfer, r_h . A method for determining r_s is described by Allen et al. (1998) and Jensen and Allen (2016). However, determination of r_h

and r_b requires additional assumption and the solution of the integral forms of the momentum transfer equation, described in Chapter 3. Considering a neutral atmospheric condition (a widely used approximation in estimating evaporation with the Penman-Monteith equation), momentum, vapor, and heat are considered to be transported equally effectively. It can thus be shown based on the flux and potential gradient relationship for vapor, heat, and momentum that

$$r_a = r_b = r_h \quad (46)$$

where r_a (s/m) is the resistance to momentum transfer, also widely referred to as the aerodynamic resistance parameter. Furthermore, combining the logarithmic wind profile equation, considered applicable to atmospheric condition in neutral stability (Penman and Long, 1960), with the steady state momentum transfer equation yields an expression for the aerodynamic resistance in terms of wind speed and surface roughness and form effects Chapter 3. The form of the r_a equation commonly used in evaporation estimation, in agricultural water management applications, is that given by Allen et al. (1998).

Based on the relationships given in Eqs. 45 and 46, the modified psychrometer constant, γ^* , in Eqs. 41, Eq. 43, and 44, can now be simplified to the form

$$\gamma^* = \gamma \left(1 + \frac{r_s}{r_a} \right) \quad (38)$$

The equations for latent heat flux, Eq. 41, sensible heat flux, Eq. 43, and the final air temperature, Eq. 44, henceforth referred to as the Penman-Monteith system of equations, cannot be evaluated directly, because the variable Δ is not known. Thus, alternative numerical solutions to this system of equations are presented in Chapter 4.

2.5. Discussion

Approaches to the derivation of the Penman-Monteith equation can fall into two categories. The most widely used approach (e.g., Jensen and Allen, 2016; Penman, 1948) considers evaporation as vapor, heat, and momentum transfer process between a source/sink surface and a point (at measurement height) in the ambient air current. The alternative approach (Monteith, 1981), on the other hand, conceptualizes evaporation as a thermodynamic process that produces changes in the thermodynamic state of the ambient air (or it describes evaporation from the vantage point of the changes it introduces into the energy state of the ambient air).

Overall, the conventional approach to the derivation of the Penman-Monteith equation has the advantage of being mathematically straight forward and compact. The thermodynamic approach, on the other hand, is more revealing of key assumptions and concepts that are generally implicit in the conventional approach. It also brings-forth some useful physical interpretations of the terms and mathematical attributes of a key parameter of the Penman-Monteith equation. The following describes the advantages accruing from the use of the thermodynamic formulation to the derivation of the Penman-Monteith equation and some additional points.

The thermodynamic method uses a two-step approach to derive the Penman-Monteith equation. As part of the mechanics of the derivation process, the initial step reveals that the Penman-Monteith equation is fundamentally a description of the transfer of vapor and heat between a wet source/sink surface and a stationary ambient air, a fact accentuated by the use of the thermodynamic equations of state to define potential difference. The dynamic effects of wind

and crop effects on evaporation are taken into account through the introduction of resistance parameters into the basic equations in the subsequent step. A step that underlines the notion that, in the Penman-Monteith equation, the convective transfer of heat, vapor, and momentum and crop-physiological effects on evaporation are taken into account in the approximate sense. Although these observations can certainly be made in the context of the conventional derivation, they are, nonetheless, implicit and hence are not as obvious and natural to the derivation as is the case with the thermodynamic approach.

An important consequence of applying the thermodynamic approach to the derivation of the Penman-Monteith equation is that it emphasizes the notion that evaporation is a process that is fundamentally driven by energy (heat) supply, although heat effects on evaporation are modulated by other weather parameters (including, vapor pressure deficit and wind speed) and crop factors. Accordingly, the thermodynamic approach shows that each term in the Penman-Monteith equation represents evaporation flux attributable to two separate sources of heat: *(i)* the sensible heat extant in the air parcel at the initial state of the air (which is a function of the thermodynamic history of the air parcel) contributes to evaporation through a process of adiabatic cooling that eventually leads to saturation of the air at the wet-bulb temperature (Figures 1 to 3). And *(ii)* a steady supply of external heat (i.e., $T_a > T_w$), in the form of net surface heat flux minus heat stored in the soil, contributes to a further increase in the latent heat content of the air while maintaining a state of saturation.

A close look at the conceptual charts (Figures 1 to 3) used in the derivation presented here, in light of the fact that the final (equilibrium) air temperature is equal to the steady surface

temperature, readily reveals that evaporation can occur with a surface temperature that is less than the air temperature, provided the surface temperature is greater than the dewpoint temperature. While such an observation can possibly be made in the context of the conventional derivation, it would, however, be intuitive and hence not as evident and inherent to the derivation as is the case with the thermodynamic approach.

Given that the thermodynamic derivation is based on conceptual charts depicting the energy states of the air parcel (Figures 1 to 3), it readily reveals the mathematical/physical attributes of a key parameter of the Penman-Monteith equation, namely: Δ . Δ is shown to be the slope of a line connecting a pair of points on the saturation vapor pressure curve. Specifically, the thermodynamic formulation readily reveals that the introduction of this concept (i.e., the Δ parameter and the line associated with it) allowed for any thermodynamic state on the saturation vapor pressure curve, say $[t, e_s(t)]$ in Figure 1, to be related to that of $[T_w, e_s(T_w)]$ with a simple, yet exact (linear) equation, provided Δ is allowed to vary with the air temperature, t . The derivation has led to an alternative form of the Penman-Monteith equation, which is expressed in terms of Δ_a and Δ_d (shorthand for $\Delta(T_0)$ and $\Delta(T_a)$, respectively). However, in the interest of keeping the resultant equation simple Δ is treated as a constant (i.e., $\Delta_a \approx \Delta_d$), contingent on the assumption that the temperature interval $[T_0, T_a]$ is sufficiently small for the accuracy of the resultant expression to be considered acceptable. Note that this concept is key to the derivation of the Penman-Monteith equation.

The derivation shows that the introduction of the air density parameter into the thermodynamic equations of state and hence into the Penman-Monteith equation, which resulted in heat contents

of the air being expressed in a unit volume basis, presumes that the range of (air) temperature variation during evaporation, i.e., $[T_a - T_w]$, is sufficiently small for the density of the air to be adequately represented by an average value. Note that this constraint on the range of temperature variation is independent of the requirement that relate to the constancy assumption of the Δ parameter.

A close look at the Penman-Monteith equation shows that both terms of the equation contain the modified psychrometer constant, γ^* , which is a function of the aerodynamic resistance parameter. The fact that both terms of the Penman-Monteith equation represent heat energy, only from different sources, and that both terms contain the parameter γ^* suggest that the thermodynamic description of the terms as adiabatic and diabatic components (Monteith and Unsworth, 2013) might be a more precise interpretation.

In the conventional approach to the derivation of the Penman-Monteith equation, potential difference is defined as the difference in energy levels between two points in space (i.e., a point in the source/sink surface and another one in the air current). By contrast, in the thermodynamic formulation, potential difference is defined as the difference in the energy levels between the thermodynamic states of a suitably defined system. Thus, this may raise the question whether the physical meaning associated with the parameters r_v and r_h in the conventional approach (as resistance parameters to the transfer of vapor and heat, respectively) and the method used to evaluate them can apply directly to the thermodynamic formulation. This point may, perhaps, need to be looked into in some detail.

Chapter 3. Integrated forms of the vapor, heat, and momentum transfer equations and the respective resistance parameters

3.1. Introduction

The derivation presented in Chapter 2 produces a small system of equations for computing the rate of transfer of vapor and heat from a source/sink surface to the ambient air and the change in air temperature during evaporation. However, another important physical process integral to evaporation and is inextricably coupled to the vapor and heat transfer processes is that of momentum transfer. Natural evaporation is generally conceptualized as a process driven mainly by near surface turbulent transfer and transport processes, caused by wind induced forced convection coupled with crop-canopy system effects on evaporation. Thus, the momentum transfer equation is key to developing expressions for the resistance parameters to vapor and heat transfer as well (which are parameters of the system of equations derived in Chapter 2). Thus, the derivation of the equation for the resistance parameter to momentum transfer (or the aerodynamic resistance) and its subsequent adaption for use in evaluating the resistance parameters to vapor and heat transfer, under suitably defined atmospheric condition, is described in the current chapter.

Accordingly, following Monteith (1981) and Monteith and Unsworth (2013), the general relationship between flux and potential gradient is used to deduce integral expressions for the resistance parameters to the convective transfer of vapor, heat, and momentum between a surface and the ambient air. Assuming natural evaporation from a cropped field takes place under neutrally stable atmospheric condition, it is shown that the aerodynamic resistance can be expressed in terms of horizontal wind velocity (a readily measurable weather parameter) and

surface roughness and form effects (both of which are related crop). Furthermore, it was noted that, under such conditions, the resistance parameters to vapor and heat transfer and the aerodynamic resistance are equal. The aerodynamic resistance parameter is then substituted for the resistance parameters to vapor and heat transfer in the Penman-Monteith equation, reducing it into a readily soluble form.

3.2. Integrated forms of the transfer equations

Fluxes associated with the exchange of vapor, heat, or momentum between a surface and the ambient air, where molecular processes dominate, can be expressed as the product of the transfer coefficient and the spatial gradient of the physical quantity of interest multiplied by the density of the air. These equations are based on Fick's law for diffusive transport of vapor, Fourier's equation for heat conduction, and Newton's law of viscosity for momentum. Equations of the same form are used to model the convective transfer, of these physical quantities, driven by turbulence, but with a different set of transfer coefficients (Monteith and Unsworth, 2013). Thus, a general form of the equations of vapor, heat, and momentum flux between the ambient air and a source/sink surface can be given as (e.g., Monteith, 1981)

$$\Phi(z) = -\rho(z)K(z) \frac{dg}{dz} \quad (1)$$

where Φ is the flux of a physical quantity (which could be vapor, heat, or momentum) at height z from the surface (W/m^2), $\rho(z)$ is the density of air at height z (Kg/m^3), $K(z)$ is the turbulent transfer coefficient for a physical quantity at height z (m^2/s), and g is the amount of a physical quantity present in a unit mass of the air at height z (J/Kg), and z (m) is above ground height.

Now, consider a constant flux, Φ , between two points, say points 1 and 2 in space (a point in a surface and in the air current at measurement height, respectively). Considering the scale of the transfer pathway that is of interest to natural evaporation processes, the variation in air density, $\rho(z)$, can be considered sufficiently small and hence it can be represented adequately by an average value, ρ (Monteith, 1981). Thus, the constant fluxes of the physical quantities across the transfer pathway can be evaluated based on the integrated form of Eq. 1.

$$\Phi = \rho \frac{\int_{G_2}^{G_1} dg}{\int_{z_1}^{z_2} \frac{dz}{K(z)}} = \rho \frac{G_1 - G_2}{\int_{z_1}^{z_2} \frac{dz}{K(z)}} \quad (2)$$

In Eq. 2, G_1 and G_2 are the amounts of a given physical quantity in a unit mass of air at points 1 and 2 (J/Kg). Taking a close look at Eq. 2, one may draw analogy between Ohm's law which relates the flux of electrical charge (current) between two points in a circuit with the quotient of the electrical potential difference between the points and the resistance across the points.

Accordingly, the numerator on the right hand-side of Eq. 2 can be viewed as the potential difference between points 1 and 2 and the denominator can be considered as the resistance to the transfer of a physical quantity across the transfer pathway, given in a more compact form as

$$\Phi = \frac{\rho \delta G}{r} \quad (3)$$

In Eq. 3, δG is the difference between G_1 and G_2 (Eq. 2). Note that if the physical quantity to be evaluated is momentum, then G represents horizontal velocity of wind, u . If, on the other hand, the physical quantity of interest is heat, then G stands for the product of heat capacity of air at constant pressure and the air temperature, $c_p T$. For the case in which the interest is to calculate

vapor flux, G represents the product of latent heat of vaporization of water and specific humidity of the air per unit mass, $\lambda\xi$.

Furthermore, in Eq. 3, r (s/m) is the resistance to the transfer of a given physical quantity between the points of interest and can be expressed as

$$\left. \begin{aligned} r_a &= \int_{z_1}^{z_2} \frac{dz}{K_m(z)}, \text{ for momentum transfer} \\ r_h &= \int_{z_1}^{z_2} \frac{dz}{K_h(z)}, \text{ for heat transfer} \\ r_b &= \int_{z_1}^{z_2} \frac{dz}{K_b(z)}, \text{ for vapor transfer} \end{aligned} \right\} \quad (4)$$

In Eq. 4, r_a is the resistance to momentum transfer (aerodynamic resistance), r_h is the resistance to heat transfer, and r_b is the resistance to the transfer of vapor across the turbulent boundary layer.

Assuming an atmospheric condition in neutral stability (a common practice in estimating evaporation with the Penman-Monteith equation), momentum, heat, and vapor are considered to be transported equally effectively (Monteith and Unsworth, 2013) and hence

$$K_m = K_h = K_b \quad (5)$$

Given the relationship in Eq. 5 and the expressions in Eq. 4, it can be readily observed that

$$r_a = r_b = r_h \quad (6)$$

The equivalence of r_a , r_h , and r_b given in Eq. 5 is important. The fact that $r_h = r_b$ will help simplify the expression for the modified psychrometer constant, γ^* , Eq. 28 of Chapter 2, leading to the form

$$\gamma^* = \gamma \left(1 + \frac{r_s}{r_h} \right) \quad (7)$$

Furthermore, because r_a can be expressed in terms of the horizontal wind speed (a readily measurable weather parameter), surface roughness, and a variable akin to a form factor (Monteith and Unsworth, 2013; Allen et al., 1998) it can be substituted for r_h in Eq. 7, resulting in the following form of γ^*

$$\gamma^* = \gamma \left(1 + \frac{r_s}{r_a} \right) \quad (8)$$

The expression for r_a will now be derived.

3.3. Derivation of an expression for the resistance parameter to momentum transfer (the aerodynamic resistance)

3.3.1. Logarithmic wind velocity profile

First, the logarithmic wind velocity profile equation will be derived, which will then be used to deduce the expression for r_a . Following Monteith and Unsworth (2013), the shearing stress, τ (N/m²), under a steady wind with a horizontal velocity function ($u(z)$, where z is the height above the ground surface) can be expressed as

$$\tau = \rho K_m(x) \frac{du}{dz} \quad (9)$$

Note that for cropped fields, the ground surface is not the effective sink for momentum, instead the virtual surface considered as the effective sink for momentum is somewhere above the ground surface, but within the crop canopy. The height of this surface from the ground is termed as the zero-plane displacement height, d . Thus, in Eq. 9, $x = z-d$, z being the height above the ground surface at which momentum flux is to be determined. Note that Eqs. 1 and 9 have the same general form except that Eq. 9 does not have a negative algebraic sign. The reason is in Eq. 1 the interest is to define the momentum flux, which has the same magnitude as the shear stress, but is vector quantity and hence have an algebraic sign. Equation 9, on the other hand, is an expression for the magnitude of the shear stress and hence no algebraic sign is attached to it.

Equation 9 can be expressed as

$$\frac{du}{dz} = \frac{\tau}{\rho K_m(x)} \quad (10)$$

A mathematically equivalent form to Eq. 10 is

$$\frac{du}{dz} = f(\tau, \rho, x) \quad (11)$$

An alternative expression for the velocity gradient can be deduced based on dimensional reasoning (e.g., Arya, 1991). Thus, following Buckingham's π theorem it can be shown that

$$\frac{du}{dz} = \frac{v}{x} \sqrt{\frac{\tau}{\rho}} \quad (12)$$

In Eq. 12, ν is a dimensionless proportionality constant [-] and is taken to be the reciprocal of a physical constant called the von Karman constant (κ), and the expression $\sqrt{\frac{\tau}{\rho}}$ is the shear velocity, u_* (m/s).

Eq. 12 can, now, be expressed in terms of the shear velocity, u_* , as follows

$$\frac{du}{dz} = \frac{u_*^2}{\kappa u_* x} \quad (13)$$

Furthermore, Eq. 10 can be written in terms of u_* as

$$\frac{du}{dz} = \frac{u_*^2}{K_m(x)} \quad (14)$$

It can be observed from Eqs. 13 and 14 that

$$K_m(x) = \kappa u_* x \quad (15)$$

Substituting the righthand side of Eq. 15 in Eq. 14 and simplifying and noting that $x = z-d$ yields

$$du = \frac{u_*}{\kappa(z-d)} dz \quad (16)$$

Integrating the lefthand side of Eq. 16 between $u = 0$ and $u = u(Z)$, (i.e., the wind velocity at height Z) and the righthand side between a lower limit of $d+z_0$ (i.e., the height at which $u = 0$) and an upper limit of Z , yields an expression for $u(Z)$

$$u(Z) = \frac{u_*}{\kappa} \ln \left(\frac{Z-d}{z_0} \right) \quad (17)$$

In Eq. 17, z_0 (m) is a constant termed as the roughness length for momentum transfer. Penman and Long (1959) reported that a logarithmic wind profile of the form given in Eq. 17 conforms to experience under neutral atmosphere. For atmospheric conditions that are not neutral, the right side of Eq. 17 need to be augmented with an additional term (e.g., Jensen and Allen, 2016).

3.3.2. Equation for the resistance to momentum transfer (aerodynamic resistance)

Following Monteith and Unsworth (2013), the derivation of the expression for the aerodynamic resistance parameter, r_a , starts with the shearing stress equation (which is the same as the gradient momentum flux equation, without the algebraic sign)

$$\tau = \rho K_m \frac{du}{dz} \quad (18)$$

Following Ohm's law analogy (Eq. 3), Eq. 18 can be expressed as ratio of the difference between the momentum in a unit volume of air at points 1 and 2 and the aerodynamic resistance between the points

$$\tau = \rho \frac{u_2 - u_1}{r_a} \quad (19)$$

Noting that $u_*^2 = \frac{\tau}{\rho}$, and rearranging terms yields an expression for the aerodynamic resistance

$$r_a = \frac{u_2 - u_1}{u_*^2} \quad (20)$$

For a scenario in which wind velocity is measured at two points, i.e., Z_1 and Z_2 , the expression for r_a can be obtained by substituting the expressions for u_2 and u_1 (Eq. 17) into Eq. 20

$$r_a = \frac{\frac{u_*}{k} \left(\ln \left(\frac{Z_2 - d}{z_0} \right) - \ln \left(\frac{Z_1 - d}{z_0} \right) \right)}{u_*^2} = \frac{\ln \left(\frac{Z_2 - d}{Z_1 - d} \right)}{ku_*} \quad (21)$$

If, on the other hand, wind speed is measured at a single point, say at height Z (as is commonly the case in practice), then the other point on the wind velocity profile can be assumed to be at the height where $u = 0$ (i.e., $d + z_0$). Equation 21 can, thus, be expressed as

$$r_a = \frac{u(Z) - u(d + z_0)}{u_*^2} \quad (22)$$

Substituting the expression for $u(Z)$, Eq. 17, into Eq. 22 and noting that $u(d + z_0) = 0$ yields

$$r_a = \frac{\ln \left(\frac{Z - d}{z_0} \right)}{ku_*} \quad (23)$$

Substituting the expression for u_* , from Eq. 17, into Eq. 23 and simplifying yields

$$r_a = \frac{\ln \left(\frac{Z - d}{z_0} \right)^2}{u(Z)k^2} \quad (24)$$

Perhaps, in recognition of the differences in roughness lengths and hence in the height of the exchange surfaces for momentum, on one hand, and that of heat and vapor, on the other, the form of the equation often used to evaluate the aerodynamic resistance in agricultural applications has the following form (e.g., Allen et al., 1998)

$$r_a = \frac{\ln\left(\frac{Z-d}{z_{0m}}\right) \ln\left(\frac{Z-d}{z_{0h}}\right)}{u(Z)k^2} \quad (25)$$

In Eq. 25, z_{0m} is the roughness height for momentum transfer (m) and z_{0h} is the roughness height for heat and vapor transfer (m).

An important note here is that in the conventional approach to the derivation of the Penman-Monteith equation, potential difference is defined as the difference in energy levels between two points in space (i.e., a point in the source/sink surface and another one in the air current). By contrast, in the thermodynamic formulation, potential difference is defined as the difference in the energy levels between the thermodynamic states of a suitably defined system. Thus, this may raise the question whether the physical meaning associated with the parameters r_a , r_h , and r_b in the conventional approach (as resistance parameters to the transfer of vapor and heat, respectively) and the method used to evaluate them can apply directly to the thermodynamic formulation. This point may, perhaps, need to be looked into in some detail.

Chapter 4. The Penman-Monteith system of equations: Numerical solutions and evaluation

4.1. Introduction

A review of the derivation of the Penman-Monteith equation based on the thermodynamic approach of Monteith (1965; 1981) is presented in Chapter 2. Evidently, from agricultural water management standpoint the most important equation that emerges from the derivation is the expression for latent heat flux, ℓ_f . However, the complete set of the Penman-Monteith system of

equations consists of the sensible heat flux, q_f , and the final air temperature, T_a , equations, in addition to that of the latent heat flux equation. Notably, this system of equations consists of three equations in four variables (ℓ_f , q_f , T_a , and a slope parameter related to the saturation vapor pressure curve, Δ), hence, it is in indeterminate form.

One of two approaches can be used to circumvent this problem. The most widely used approach, referred here as the conventional model, determines Δ independently, based on the assumption that a satisfactory approximation of Δ can be obtained if it is set to the slope of the saturation vapor pressure curve at the measured air temperature, T_0 (Allen et al., 1998; Jensen and Allen, 2016). The Δ value determined as such is then substituted in the latent heat flux, sensible heat flux, and surface temperature equations to calculate ℓ_f , q_f , and T_a directly.

An alternative approach described by McArthur (1990), and expanded in the current study, involves augmenting the ℓ_f , q_f , and T_a equations with an approximate expression that is obtained based on the constancy assumption of Δ described in Chapter 2. Note that for brevity, this equation will henceforth be referred to as the Δ equation. The resultant set consisting of four equations in four variables (ℓ_f , q_f , T_a , and Δ) can then be solved numerically. McArthur (1990) solved the latent heat flux, ℓ_f , surface temperature, and Δ equations using a sequential function-evaluation iterative scheme. Jensen and Allen (2016) also highlighted iterative solution of ℓ_f , q_f , the surface temperature, and the Δ equations.

A close look at the mathematical structure of the equations suggest that alternative numerical solutions ranging in complexity from those involving the solution of a subset of the system of equations to those solving the complete set can be developed. Accordingly, four alternative algorithms (referred to as model 1, 2, 3, and 4) were developed and evaluated in the current study: (i) Programmatic implementation of McArthur's approach, which uses a

sequential function-evaluation based iterative scheme to determine T_a and Δ followed by direct calculation of ℓ_f and q_f (described here as model 1). (ii) Simultaneous iterative solution of the final air temperature and Δ equations for T_a and Δ , followed by direct calculation of ℓ_f and q_f (labeled as model 2). (iii) Newton iteration of a nonlinear equation in T_a , obtained by combining the T_a and Δ equations in a way that eliminates Δ . Followed by direct calculation of Δ , ℓ_f , and q_f (referred here as model 3). And (iv) Simultaneous iterative solution of a form of the complete set of equations obtained by recasting each of the latent heat flux, sensible heat flux, and the final air temperature equations into a form where the variables are ℓ_f , q_f , and T_a only (described as model 4).

Results of model verification, conducted based on data reported by McArthur (1990), showed that each of the alternative models produced outputs that are essentially identical and also in close agreement with the solution reported by McArthur, a study considered here as a benchmark against which the performance of the current models are referenced. The mean absolute residuals between variable estimates computed with the models presented here and those reported by McArthur range from a minimum of about 0.2% for T_a and a maximum of 1.9% for Δ . Furthermore, intercomparison of the alternative models based on the criteria of numerical efficiency and robustness revealed that each model represents a comparable alternative, to each of the other models, for estimating evaporation. However, model 1 (which is based on sequential function-evaluation approach) is simpler than model 2 (which is based on a simultaneous iterative solution of a form of the complete set) both conceptually and in terms of its numerical formulation and programmatic implementation. Thus, model 1 is considered here for further use.

The performance of model 1 is then compared against those of the conventional model based on a total of seven hypothetical data sets, covering a wide range of natural evaporation conditions. The results suggest that differences in the methods used for estimating Δ have the maximum effect on sensible heat flux estimates (where the mean absolute residual is 18.1%), a negligible effect on estimates of the final air temperature (with an average residual of 0.7%), and a limited effect on the estimates of latent heat flux (where the mean residual is 8.2%).

Given that both model 1 and the conventional model involve a level of approximation in the determination of Δ , it can be observed that direct comparison of the two models cannot provide answer to the question: which model is more accurate? Thus, the current study does not try to answer that question. However, considering the latent heat flux, ℓ_f (which is the practically more important of the evaporation variables), the relatively small average residual (of 8.2%) suggests that the margins of error of the ℓ_f estimates obtained with model 1 and the conventional model may not be significantly different. This suggests that, at least for a good set of applications, the use of both the conventional model and model 1 can be considered equally acceptable from the standpoint of accuracy. However, the conventional model is preferred, because of its simplicity. Nonetheless, it ought to be emphasized that the study presented here is based on a limited data and hence a conclusive deduction needs to await a more comprehensive follow up study.

4.2. Equations

From the standpoint of agricultural water management the most important equation that emerges from the derivation of the Penman-Monteith equation is, evidently, the equation for latent heat

flux, ℓ_f . However, the complete set of the Penman-Monteith system of equations consists of the sensible heat flux, q_f , and the final air temperature, T_a , equations, in addition to that of the latent heat flux equation

$$\ell_f = \frac{\Delta Q_f}{\Delta + \gamma^*} + \rho c_p \frac{e_s(T_0) - e_0}{(\Delta + \gamma^*)r_a}, \quad \text{where } \ell_f = \lambda E_f \text{ and } \gamma^* = \gamma \left(1 + \frac{r_s}{r_a}\right) \quad (1)$$

$$q_f = \frac{\gamma^* Q_f}{\Delta + \gamma^*} - \rho c_p \frac{e_s(T_0) - e_0}{(\Delta + \gamma^*)r_a} \quad (2)$$

$$T_a = T_0 + \frac{\gamma^* r_a Q_f}{\rho c_p (\Delta + \gamma^*)} - \frac{e_s(T_0) - e_0}{\Delta + \gamma^*} \quad (3)$$

In Eqs. 1 to 3, ℓ_f (W/m²) is latent heat flux; λ (J/Kg) is the latent heat of vaporization of water; E_f (Kg/s-m²) is evaporation flux; Δ (KPa/°K) is slope parameter related to the saturation vapor pressure function; Q_f (W/m²) is the available energy for partitioning into sensible and latent heat components (i.e., the external heat flux into the air, which is equal to the net surface heat flux minus the ground heat flux); γ^* (KPa/°K) is the modified psychrometer constant; γ (KPa/°K) is the psychrometer constant; r_s (s/m) is bulk surface resistance; r_a (s/m) aerodynamic resistance; ρ (Kg/m³) is the density of the ambient air; c_p (J/Kg-°K) is the heat capacity of the air at constant pressure; $e_s(T)$ (KPa) is the saturation vapor pressure at temperature, T ; T_0 (°K) is the measured air temperature; e_0 (KPa) is the measured vapor pressure; and T_a (°K) is the final air temperature.

In Eqs. 1 to 3, the latent heat flux, ℓ_f , the slope parameter related the saturation vapor pressure curve, Δ , the sensible heat flux, q_f , and the final air temperature, T_a , are variables and the rest of the physical quantities are given parameters. Notably, this system of equations contains three equations with four variables, which implies that it is of indeterminate form. Thus,

in order to obtain a unique solution, the system must be conditioned following either one of the following approaches: one of the variables needs to be determined independently of Eqs. 1 to 3 or Eqs. 1 to 3 need to be augmented with an additional equation. Generally, the following two approaches are used to solve this problem, both involving a level of approximation in the determination of Δ .

The most widely used approach, referred here as the conventional model, assumes that a satisfactory approximation of Δ can be obtained if it is set to the slope of the saturation vapor pressure function at the measured air temperature, $\left. \frac{\partial e_s}{\partial T} \right|_{T=T_0}$, (Allen et al., 1998; Jensen and Allen, 2016; Howell and Evett, 2004). The assumption is predicated on the consideration that the difference between the surface and the air temperatures are generally sufficiently small for $\left. \frac{\partial e_s}{\partial T} \right|_{T=T_0}$ to be a good approximation of Δ and for the resultant estimate of ℓ_f to be of acceptable accuracy. The Δ value determined, independently, is then substituted into Eqs. 1, 2, and 3 to calculate ℓ_f , q_f , and T_a , respectively.

The second approach involves augmenting Eqs. 1 to 3 with an additional equation of the form

$$e_s(T_0) = e_s(T_a) - \Delta(T_a - T_0) \quad (4)$$

Note that Eq. 4 can be deduced from Eq. 12 of Chapter 2 based on the constancy assumption of Δ .

Equations 1 to 4 consist of a system of four equations with four variables: ℓ_f , q_f , T_a , and Δ . Note that the exact form of Eq. 4 requires Δ to be variable with the air temperature, which would have resulted in a pair of equations, wherein one of the equations is expressed in terms of,

$\Delta_a = \Delta(T_0)$, and another one given in term of, $\Delta_d = \Delta(T_a)$. However, if the temperature range $[T_0, T_a]$ can be considered sufficiently small, then Δ can be treated as a constant (e.g., Monteith, 1981). Once Δ is assumed to be a constant, then efficient and robust iterative solutions can be readily developed for Eqs. 1 to 4. A form of Eq. 4 expressed in terms of the surface temperature was used by McArthur (1990) to compute Δ . As shown in Chapter 2, the surface temperature is the same as the final (equilibrium) air temperature.

A simultaneous solution of the complete set of equations (Eqs. 1 to 4 or modified forms thereof) should in principle lead to a more accurate solution. However, a close look at Eqs. 1 and 2 reveals that these equations contain one variable each (ℓ_f in the case of Eq. 1 and q_f in the case of Eq. 2) that do not appear in any of the other equations (i.e., Eqs. 3 and 4). Furthermore, Eqs. 1 and 2 share only the variable Δ with the other equations. This observation suggests that in their current form, Eqs. 1 and 2 are not as strongly coupled to Eqs 3 and 4, as for instance Eq. 3 is to that of Eq. 4 or vice-versa (where both equations are functions of T_a and Δ). The preceding observation suggests that a simpler formulation, than a simultaneous solution of the complete set of equations, could lead to sufficiently accurate but more efficient and robust solutions.

Accordingly, in the current development alternative solution techniques with a varying degree of complexity were developed and evaluated in the current study: (i) Programmatic implementation of McArthur's sequential function-evaluation based iterative scheme, in which Eqs. 3 and 4 are first solved for T_a and Δ and then the Δ value obtained as such is substituted into Eqs. 1 and 2 to determine ℓ_f and q_f (referred here as model 1). (ii) Simultaneous iterative solution of Eqs. 3 and 4 for T_a and Δ with the Newton method, followed by calculation of ℓ_f and q_f with Eqs. 1 and 2 (labeled as model 2). (iii) Newton iteration of a nonlinear equation in T_a , obtained by combining Eqs. 3 and 4 in a way that eliminates Δ . The T_a value computed as such is then

substituted in Eq. 4 to determine Δ , which in turn is substituted into Eqs. 1 and 2 to calculate ℓ_f and q_f (described as model 3). And (iv) Simultaneous iterative solution of a form of the complete set of equations obtained by recasting Eqs. 1 to 3 into a form where the variables are ℓ_f , q_f , and T_a only (referred here as model 4). Note that the modified set of equations are obtained through substitution of an expression for Δ deduced based on Eq. 4.

Models 1, 2, and 3 use a two-step approach to determine the variables: ℓ_f , q_f , T_a , and Δ . The first phase uses different formulations/iteration methods to solve Eqs. 3 and 4 for T_a and Δ . Then, in the second phase, the Δ value so computed is substituted into Eqs. 1 and 2 to determine ℓ_f and q_f , respectively. Model 4, on the other hand, computes the variables ℓ_f , q_f , and T_a through a simultaneous solution of a form Eqs. 1 to 4.

4.3. Description of the numerical algorithms

4.3.1. Function evaluation based iterative scheme of McArthur (1990), model 1

A numerical algorithm developed, in the current study, for the programmatic implementation of McArthur's function-evaluation based procedure is described here.

4.3.1.a. Computational procedure

1. Initialize the iteration index and variables:

(1a) Set the iteration index, i , to 0. Go to step 1b.

(1b) Set $T_a^i = 1.05T_0$, where T_a^i is the i th estimate of T_a . Go to step 2.

2. Calculate the i th estimate of Δ , Δ^i , with Eq. 5 (deduced from Eq. 4)

$$\Delta^i = \frac{e_s(T_a^i) - e_s(T_0)}{T_a^i - T_0} \quad (5)$$

where

$$e_s(T) = 0.611 \exp\left(\frac{17.27(T-273)}{T-36}\right) \quad (6)$$

In Eq. 6, T is air temperature ($^{\circ}\text{K}$). Go to step 3.

3. Calculate a revised estimate of T_a , $T_a^{(i+1)}$, with Eq. 7 (from Eq. 3)

$$T_a^{(i+1)} = T_0 + \frac{\gamma^* r_a Q_f}{\rho c_p (\Delta^i + \gamma^*)} - \frac{e_s(T_0) - e_0}{\Delta^i + \gamma^*} \quad (7)$$

Go to step 4.

4. If $i = 0$, then go to step 8. If, on the other and, $i > 0$, then go to step 5.
5. Calculate the incremental changes on T_a and Δ :

(5a) Calculate the $(i+1)$ th incremental change in T_a , $\delta T_a^{(i+1)}$, with Eq. 8

$$\delta T_a^{(i+1)} = T_a^{(i+1)} - T_a^i \quad (8)$$

Go to step 5b.

(5b) Calculate the incremental change in Δ at the i th iteration, $\delta \Delta^i$, with Eq. 9

$$\delta \Delta^i = \Delta^i - \Delta^{(i-1)} \quad (9)$$

Go to step 6.

6. Convergence test

(6a) If $\delta T_a^{(i+1)} \leq 0.001\text{s}$, then go to step 6b. If, on the other hand, $\delta T_a^{(i+1)} > 0.001\text{s}$, then go to step 8.

(6b) If $\delta\Delta^i \leq 0.00001\text{KPa}/^\circ\text{K}$, then go to step 7. If, on the other hand, $\delta\Delta^{(i-1)} >$

$0.00001\text{KPa}/^\circ\text{K}$, then go to step 8.

7. The iteration has converged and the solution is : $T_a = T_a^i$ and $\Delta = \Delta^i$. Go to step 11.
8. Set $i = i+1$ and go to step 9.
9. If $i \leq \text{MaxIteration}$ (where *MaxIteration* is the maximum number of iterations allowed), then go to step 2. If, on the other hand, $i > \text{MaxIteration}$ then go to step 10.
10. The iteration failed to converge. Go to step 11.
11. End computation.

II. Calculate the latent heat flux, ℓ_f , evaporation flux, E_f , and the sensible heat flux, q_f :

1. Calculate ℓ_f and q_f with Eqs. 1 and 2, respectively, based Δ values computed in phase I, above.
2. Calculate the evaporation flux, $E_f(\text{mm/day})$ with

$$E_f = 3.525 \times 10^{-2} \ell_f \quad (10)$$

4.3.2. Simultaneous solution of equations 3 and 4 with the Newton method, model 2

4.3.2.a. Equations

With this approach, Eqs. 3 and 4 are solved simultaneously with the Newton method.

Accordingly, Eqs. 3 and 4 are first written in the form, $F(\mathbf{x}) = 0$, where $\mathbf{x}$ is the variable vector

$$0 = T_a - T_0 - \frac{\gamma^* r_a Q_f}{\rho c_p (\Delta + \gamma^*)} + \frac{e_s(T_0) - e_0}{\Delta + \gamma^*} \quad (11)$$

$$0 = e_s(T_a) - e_s(T_0) - \Delta(T_a - T_0) \quad (12)$$

At each Newton iteration, say at the i th iteration, a pair of linear equations of the form given in Eq. 13 are solved simultaneously

$$\mathbf{A}(\mathbf{x}^i) \delta \mathbf{x}^{(i+1)} = -\mathbf{F}(\mathbf{x}^i) \quad (13)$$

In Eq. 13, $\mathbf{x}^i$ is the variable vector at the i th iteration, given as

$$\mathbf{x}^i = \begin{pmatrix} T_a^i \\ \Delta^i \end{pmatrix} \quad (14)$$

$\mathbf{F}$ is the residual vector whose elements are the right-hand sides of Eqs. 11 and 12 evaluated at $\mathbf{x} = \mathbf{x}^i$

$$\mathbf{F}(\mathbf{x}^i) = \begin{pmatrix} F_{11}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i} \\ F_{12}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i} \end{pmatrix} \quad (15)$$

In Eq. 15, $F_{11}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}$ and $F_{12}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}$ are the righthand sides of Eqs. 11 and 12 evaluated at $\mathbf{x} = \mathbf{x}^i$. $\mathbf{A}$ is a coefficient matrix given as

$$\mathbf{A}(\mathbf{x}^i) = \begin{pmatrix} [\nabla F_{11}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T \\ [\nabla F_{12}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T \end{pmatrix} \quad (16)$$

where $[\nabla F_{11}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T$ and $[\nabla F_{12}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T$ are the transposes of the gradient vectors of the right-hand sides of Eqs. 11 and 12, respectively, evaluated at $\mathbf{x} = \mathbf{x}^i$ and are expressed as

$$[\nabla F_{11}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T = \left(1.0, -\frac{e_s(T_0)-e_0-\frac{\gamma^* r_a Q_f}{\rho c p}}{(\Delta^i + \gamma^*)^2} \right) \quad (17)$$

$$[\nabla F_{12}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T = \left(4093.0 \frac{e_s(T_a^i)}{(T_a^i - 36)^2} - \Delta^i, -(T_a^i - T_0) \right) \quad (18)$$

Furthermore, in Eq. 13, $\delta \mathbf{x}^{(i+1)}$ is the $(i+1)th$ incremental change in the variable vector

$$\delta \mathbf{x}^{(i+1)} = \begin{pmatrix} \delta T_a^{(i+1)} \\ \delta \Delta^{(i+1)} \end{pmatrix} \quad (19)$$

4.3.2.b. Computational procedure, model 2

I. Iterative algorithm for computing T_a and Δ , model 2

1. Initialize the iteration index, i , and the variables T_a and Δ
 - (1a) Set $i = 0$. Go to step 1b.
 - (1b) $T_a^i = 1.05T_0$. Go to step 1c.
 - (1c) Calculate Δ^i with Eq. 5. Go to step 2.
2. Evaluate elements of the function vector, F_{11} and F_{12} at $\mathbf{x} = \mathbf{x}^i$, with the right-hand side expressions of Eqs. 11 and 12. Go to step 3.
3. Calculate elements of the coefficient matrix, A^i , at $\mathbf{x} = \mathbf{x}^i$, with Eqs. 17 and 18. Go to step 4.
4. Calculate the $(i+1)th$ incremental change in the variable vector, $\delta \mathbf{x}^{(i+1)}$, by solving Eq. 13 with the Cramer's rule (e.g., Lipschutz, 1991). Go to step 5.
5. Convergence test:
 - 5a. If $\delta T_a^{(i+1)} \leq 0.001s$, then go to step 5b. On the other hand, if $\delta T_a^{(i+1)} > 0.001s$, then go to step 7.

5b. If $\delta\Delta^{(i+1)} \leq 0.00001 \text{ KPa}^\circ\text{K}$, then go to step 6. If, on the other hand, $\delta\Delta^{(i+1)} >$

$0.00001 \text{ KPa}^\circ\text{K}$, then go to step 7.

6. The iteration has converged and the solution is : $T_a = T_a^i$ and $\Delta = \Delta^i$. Go to step 11.

7. Update variables:

$$(7a) T_a^{(i+1)} = T_a^i + \delta T_a^{(i+1)} \quad (20)$$

Go to step 7b.

$$(7b) \Delta^{(i+1)} = \Delta^i + \delta\Delta^{(i+1)} \quad (21)$$

Go to step 8.

8. Set $i = i+1$, proceed to step 9.

9. If $i \leq \text{MaxIteration}$, then go to step 2. If, on the other hand, $i > \text{MaxIteration}$ then go to step 10.

10. Iterative solution of Eqs. 3 and 4 for T_a and Δ failed to converge. Go to step 11.

11. End computation.

II. Calculate ℓ_f , E_f , and q_f following the steps described under model 1, phase II

4.3.3. Iterative solution of the combined equation with the Newton method, model 3

4.3.3.a. Equations

With this approach, Eqs. 3 and 4 are combined into a single nonlinear equation with T_a as the variable

$$0 = T_a - T_0 - \frac{\gamma^* r_a Q_f \zeta}{\rho c_p (\phi + \gamma^* \zeta)} + \zeta \frac{e_s(T_0) - e_0}{\phi + \gamma^* \zeta} \quad (22)$$

Equation 22 is obtained by substituting an expression for Δ

$$\Delta = \frac{\phi}{\varsigma} \quad (23)$$

into Eq. 3. Thus, in Eq. 23

$$\phi = e_a(T_a) - e_s(T_0) \quad \text{and} \quad \varsigma = T_a - T_0 \quad (24)$$

Noting that ϕ and ς are both functions of T_a , it can be observed that Eq.22 is a nonlinear equation in T_a and is solved here iteratively with the Newton method. Accordingly, in each iterative step, say at the i th iteration, the $(i+1)$ th incremental change in T_a , $\delta T_a^{(i+1)}$, is calculated with Eq. 25

$$\delta T_a^{(i+1)} = - \frac{F(T_a^i)}{\left. \frac{dF(T_a)}{dT_a} \right|_{T_a=T_a^i}} \quad (25)$$

where $F(T_a)$ is the right-hand side of Eq. 22 evaluated at $T_a = T_a^{(i+1)}$ and

$$\left. \frac{dF(T_a)}{dT_a} \right|_{T_a=T_a^i} = 1.0 - \left(\frac{\gamma^* r_a Q_f}{\rho c_p} - e_s(T_0) + e_0 \right) \frac{\left(\phi^i - 4093.0 \varsigma^i \frac{e_s(T_a^i)}{(T_a^i - 36.0)^2} \right)}{(\phi^i + \gamma^* \varsigma^i)^2} \quad (26)$$

4.3.3.b. Computational procedure, model 3

I. Iterative algorithm for computing T_a , model 3

1. Initialize the iteration index, i , and variable, T_a .

(1a) Set the iteration index, i , to 0. Go to step 1b.

(1b) Set $T_a^i = 1.05T_0$. Go to step 2.

2. Calculate $F(T_a^i)$ with the righthand side expression of Eq. 22. Go to step 3.

3. Calculate $(dF(T_a)/dT_a)|_{T_a=T_a^i}$ with Eq. 26. Go to step 4.

4. Calculate the $(i+1)th$ incremental change in the variable vector, $\delta T_a^{(i+1)}$, with Eq. 25. Go to step 5.
5. Convergence test: If $\delta T_a^{(i+1)} \leq 0.001s$, then go to step 6. On the other hand, if $\delta T_a^{(i+1)} > 0.001s$, then go to step 7.
6. The iteration has converged and the solution is $T_a = T_a^i$. Go to step 11.
7. Update variable: set $T_a^{(i+1)} = T_a^i + \delta T_a^{(i+1)}$. Go to step 8.
8. Set $i = i+1$, proceed to step 9.
9. If $i \leq MaxIteration$, then go to step 2. If, on the other hand, $i > MaxIteration$ then go to step 10.
10. Iterative solution of Eq. 22 failed to converge. Go to step 11.
11. End computation.

II. Calculate the slope parameter related to the saturation vapor pressure, Δ , ℓ_f , E_f , and q_f :

1. Calculate Δ with Eq. 5 as a function T_a computed in phase I.
2. Calculate ℓ_f , E_f , and q_f following the steps described under model 1, phase II.

4.3.4. Simultaneous iterative solution of the modified forms of Eqs. 1 to 3 with the Newton method, model 4

4.3.4.a. Equations

With this approach, a system consisting of modified forms of Eqs. 1 to 3 are solved simultaneously with the Newton method. The modified equations given in the form of $F(\mathbf{x}) = 0$ are

$$0 = \ell_f - \frac{\phi Q_f}{\phi + \gamma^* \varsigma} - \rho c_p \varsigma \frac{e_s(T_0) - e_0}{(\phi + \gamma^* \varsigma) r_a} \quad (27)$$

$$0 = q_f - \frac{\varsigma \gamma^* Q_f}{\phi + \gamma^* \varsigma} + \rho c_p \varsigma \frac{e_s(T_0) - e_0}{(\phi + \gamma^* \varsigma) r_a} \quad (28)$$

$$0 = T_a - T_0 - \frac{\varsigma \gamma^* r_a Q_f}{\rho c_p (\phi + \gamma^* \varsigma)} + \varsigma \frac{e_s(T_0) - e_0}{\phi + \gamma^* \varsigma} \quad (29)$$

Note that Eqs. 27 to 29 are obtained by substituting an expression for Δ (Eq. 23) into Eqs. 1 to 3.

Considering that ϕ and ς (Eq. 24) are functions of T_a , it can be observed that Eqs. 27 to 29 constitute a nonlinear system of equations in the variables ℓ_f , q_f , and T_a . Accordingly, in each Newton iteration, say in the $(i+1)th$ iteration, a system of linear equations of the form given in Eq. 13 is solved simultaneously. However, compared to model 2, which involves a pair of equations with two variables, the system applicable to model 4 consists of three equations with three variables. Thus, the corresponding residuals, $\mathbf{F}(\mathbf{x}^i)$, and variable, $\mathbf{x}^i$, vectors are given as

$$\mathbf{F}(\mathbf{x}^i) = \begin{pmatrix} F_{27}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i} \\ F_{28}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i} \\ F_{29}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i} \end{pmatrix} \quad \text{and} \quad \mathbf{x}^i = \begin{pmatrix} \ell_f^i \\ q_f^i \\ T_a^i \end{pmatrix} \quad (30)$$

where $F_{27}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}$, $F_{28}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}$, and $F_{29}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}$ are the expressions on the righthand sides of Eqs. 27, 28, and 29, respectively, evaluated at $\mathbf{x} = \mathbf{x}^i$ and ℓ_f^i , q_f^i , and T_a^i are the latent heat, sensible heat, and final air temperature values at the i th iteration, respectively.

The coefficient matrix $\mathbf{A}$ is given as

$$\Lambda(\mathbf{x}^i) = \begin{pmatrix} [\nabla F_{27}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T \\ [\nabla F_{28}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T \\ [\nabla F_{29}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T \end{pmatrix} \quad (31)$$

where $[\nabla F_{27}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T$, $[\nabla F_{28}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T$, and $[\nabla F_{29}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T$ are the transposes of the gradient vectors of Eqs. 27, 28, and 29, respectively, evaluated at $\mathbf{x} = \mathbf{x}^i$, and are expressed as

$$[\nabla F_{27}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T = \begin{pmatrix} \left(4093.0 \zeta^i \frac{e_s(T_a^i)}{(T_a^i - 36.0)^2} \right) \left(\frac{\rho c_p}{r_a} (e_s(T_0) - e_0) - \gamma^* Q_f \right) \\ 1.0, \quad 0.0, \quad \frac{+\phi \left(\gamma^* Q_f - \frac{\rho c_p}{r_a} (e_s(T_0) - e_0) \right)}{(\phi^i + \gamma^* \zeta^i)^2} \end{pmatrix} \quad (32)$$

$$[\nabla F_{28}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T = \begin{pmatrix} 0.0, \quad 1.0, \quad \left(\frac{\rho c_p}{r_a} (e_s(T_0) - e_0) - \gamma^* Q_f \right) \frac{\left(\phi^i - 4093.0 \zeta^i \frac{e_s(T_a^i)}{(T_a^i - 36.0)^2} \right)}{(\phi^i + \gamma^* \zeta^i)^2} \end{pmatrix} \quad (33)$$

$$[\nabla F_{29}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^i}]^T = \begin{pmatrix} 0.0, \quad 0.0, \quad 1.0 + \left(e_s(T_0) - e_0 - \frac{\gamma^* r_a Q_f}{\rho c_p} \right) \frac{\left(\phi^i - 4093.0 \zeta^i \frac{e_s(T_a^i)}{(T_a^i - 36.0)^2} \right)}{(\phi^i + \gamma^* \zeta^i)^2} \end{pmatrix} \quad (34)$$

Furthermore, $\delta \mathbf{x}^{(i+1)}$ is the $(i+1)th$ incremental change in the variable vector, given as

$$\delta \mathbf{x}^{(i+1)} = \begin{pmatrix} \delta \ell_f^{(i+1)} \\ \delta q_f^{(i+1)} \\ \delta T_a^{(i+1)} \end{pmatrix} \quad (35)$$

where $\delta\ell_f^{(i+1)}$, $\delta q_f^{(i+1)}$, and $\delta T_a^{(i+1)}$ are the $(i+1)th$ incremental change in the latent heat, sensible heat, and final air temperature, respectively.

4.3.4.b. Iterative algorithm for computing T_a , ℓ_f , and q_f , model 4

1. Initialize the iteration index, i , and variables
 - (1a) Set $i = 0$. Go to step 1b.
 - (1b) Set $T_a^i = 1.05T_0$. Go to step 1c.
 - (1c) Set $\ell_f^i = \ell_f(T_a^i)$, Eq. 27. Go to step 1d.
 - (1d) Set $q_f^i = q_f(T_a^i)$, Eq. 28. Go to step 2.
2. Evaluate elements of the function vector, Eq. 30, at $\mathbf{x} = \mathbf{x}^i$. Go to step 3.
3. Calculate elements of the coefficient matrix, $\mathbf{A}^i$, at $\mathbf{x} = \mathbf{x}^i$, with Eqs. 32 to 34. Go to step 4.
4. Calculate the $(i+1)th$ incremental change in the variable vector, $\delta\mathbf{x}^{(i+1)}$, by solving a form of Eq. 13 applicable to model 4 with the Cramer's rule (e.g., Lipschutz, 1991). Go to step 5.
5. Convergence test:
 - (5a) If $\delta T_a^{(i+1)} \leq 0.001s$, then go to step 5b. On the other hand, if $\delta T_a^{(i+1)} > 0.001s$, then go to step 8.
 - (5b) If $\delta\ell_f^{(i+1)} \leq 0.001W/m^2$, then go to step 5c. If, on the other hand, $\delta\ell_f^{(i+1)} > 0.001W/m^2$, then go to step 8.
 - (5c) If $\delta q_f^{(i+1)} \leq 0.001W/m^2$, then go to step 6. If, on the other hand, $\delta q_f^{(i+1)} > 0.001W/m^2$, then go to step 8.
6. The iteration has converged and the solution is: $T_a = T_a^i$, $\ell_f = \ell_f^i$, and $q_f = q_f^i$. Go to step 7.
7. Calculate:
 - (7a) Δ with Eq. 5 based on the T_a value given in step 6. Go to step 7b.

(7b) E_f with Eq. 10. Go to step 12.

8. Update variables

$$T_a^{(i+1)} = T_a^i + \delta T_a^{(i+1)} \quad (36)$$

$$\ell_f^{(i+1)} = \ell_f^i + \delta \ell_f^{(i+1)} \quad (37)$$

$$q_f^{(i+1)} = q_f^i + \delta q_f^{(i+1)} \quad (38)$$

Go to step 9.

9. Set $i = i+1$, proceed to step 10.

10. If $i \leq \text{MaxIteration}$, then go to step 2. If, on the other hand, $i > \text{MaxIteration}$ then go to step 11.

11. Iterative solution of Eqs. 27 to 29 failed to converge. Go to step 12.

12. End computation.

4.4. Evaluation of the alternative numerical solutions for the Penman-Monteith system of equations

Evaluation of the alternative numerical solutions, presented earlier, is conducted in three steps.

The first step consists of model verification and it involves comparing variable estimates obtained with each of the models (i.e., model 1, 2, 3, and 4) with those reported by McArthur (1990). In other words, variable estimates presented by McArthur are considered here as benchmarks with respect to which variable estimates obtained with the alternative models are referenced. Models that are deemed sufficiently accurate are then carried-forward to the next step. If more than one models are considered to be of satisfactory accuracy, intercomparison of

models, using additional data sets, is conducted based on the criteria of numerical efficiency, robustness, and simplicity and the model deemed most advantageous is recommended for further consideration. In the final step, variable estimates obtained with the model selected for further use in the preceding step is compared with those computed with the conventional model and the advantages and limitations of each model are discussed.

4.4.1. Data description

A total of seven hypothetical data sets, summarized in Table 1, are used in the evaluation of the proposed models (i.e., model 1, 2, 3, and 4). Three data sets, namely data sets 1, 2, and 3, were obtained from the study reported by McArthur (1990). McArthur's data sets consider a wet surface with canopy resistance of 0s/m, resistance to heat transfer, r_h , of 100s/m (assumed here to be equivalent to the aerodynamic resistance, r_a), a saturated air ($e_s(T_0) - e_0 = 0$), a relatively strong sunshine with net external heat flux into the air of 500W/m². Although the vapor pressure, e_0 , vary from a data set to another (Table 1), the rather important parameter of vapor pressure deficit (in terms of its effect on evaporation) is zero for all the data sets. Thus, the only variable that differentiates a data set from another is the measured air temperature, which was set to 273°K for data set 1, 293°K for data set 2, and 313°K for data set 3. Note that the canopy resistance parameter is assumed here to be equivalent to the bulk surface resistance parameter, r_s , of the FAO Penman-Monteith equation (Allen et al., 1998).

Table 1. Weather, crop, and related evaporation parameters

Variable/parameter name	Unit	McArthur's data sets			Additional data sets			
		Data set 1	Data set 2	Data set 3	Data set 4	Data set 5	Data set 6	Data Set 7
Measured air temperature, T_0	°K	273	293	313	278	303	308	293
Psychrometer constant, γ	KPa/°K		0.066		0.066	0.066	0.066	0.066
External heat flux, Q_f	W/m ²		500		300	420	650	400
Air density, ρ	Kg/m ³		1.204		1.204	1.204	1.204	1.204
Heat capacity of air at constant pressure, c_p	J/Kg		1005		1005	1005	1005	1005
Vapor pressure, $e_0^{(*)}$	KPa	0.611	2.343	7.398	0.5	3.504	5.2	0.243
Wind velocity measurement height, $z_m^{(**)}$	m		-		2	2	2.5	2
Wind speed at measurement, $u(z)^{(**)}$	m/s		-		1	1.2	3.5	3
Vapor pressure measurement height, $z_h^{(**)}$	m		-		2	2	2.5	2
Roughness length, momentum transfer, $z_{om}^{(**)}$	m		-		0.0738	0.0148	0.0431	0.0923
Roughness length, heat and vapor transfer, $z_{oh}^{(**)}$	m		-		0.0074	0.0015	0.0043	0.0092
Zero plane displacement height, $d^{(**)}$	m		-		0.400	0.080	0.233	0.500
Crop height, $h^{(**)}$	m		-		0.6	0.12	0.35	0.75
Bulk surface resistance, r_s	s/m		0		40	100	120	60
Aerodynamic resistance, $r_a^{(***)}$	s/m		100		-	-	-	-

(*) For McArthur's data sets, the vapor pressures correspond to saturated conditions for the respective temperatures.

(**) For McArthur's data sets (i.e., data sets 1 to 3) the resistance to heat transfer (which is assumed here to be equivalent to the aerodynamic resistance) is set to 100s/m. Thus, the weather, crop, and other parameters required for the determination of the aerodynamic resistance are not specified.

(***) For the additional data sets (data sets 4 to 7) the aerodynamic resistance is not specified at the input, because it is calculated as a function of the weather, crop, and other parameters specified at the input

In addition, Table 1 presents four more data sets, labeled as additional data sets, covering a wide range of hypothetical evaporation scenarios from a cropped field. For these data sets, vapor pressure deficits range between 0.373 to 2.1KPa, which represent relatively humid and

rather dry air conditions, respectively. The measured air temperature ranges from 278 to 308°K and the external heat flux vary between 300 and 650W/m². Note that for the additional data sets, the aerodynamic resistance values are not specified in Table 1, because they are not inputs to the model. Instead, they were computed as a function of wind speed and roughness lengths and zero plane displacement height, both calculated as a function of crop height (Table 1) following the approach described by Allen et al. (1998). The computed aerodynamic resistance vary between a minimum of 28.2s/m (data set 7) and a maximum of 173.1s/m (data set 5).

4.4.2. Verification of models

4.4.2.a. Summary of results reported by McArthur (1990)

As noted earlier, McArthur (1990) evaluated the effects of computing Δ , with the function-evaluation based iterative method and the conventional approach, on the estimates of latent heat flux, ℓ_f , and final air temperature, T_a . The results reported by McArthur (1990) show that the differences in the estimates of ℓ_f obtained with the iterative approach and those calculated with the conventional model vary between about 1.5% (data set 3) and 40.0% (data set 1) of the values obtained with the conventional approach. Although not reported by McArthur, it can be shown that the residuals between estimates of T_a presented by McArthur and those of the conventional model range from 0.3% (data set 3) to 1.9% (data set 1) of those calculated with the conventional model. The corresponding differences between estimates of Δ obtained with the two methods vary between 12.0% (data set 3) and 79.3% (data set 1) of those computed with the conventional approach.

4.4.2.b. Comparison of variable estimates obtained with the alternative models developed here and those reported by McArthur

The computational procedures newly proposed here (i.e., model 2, 3, and 4) and the programmatic implementation of McArthur's approach (referred here as model 1) are evaluated by comparing estimates of ℓ_f , q_f , E_f , T_a , and Δ computed with each of the models with those reported by McArthur (1990). The ℓ_f , q_f , E_f , T_a , and Δ estimates obtained with each model and those reported by McArthur along with a summary of the percent absolute residuals between the estimates obtained with each of the models presented here and those reported by McArthur are shown in Tables 2a and 2b. Although McArthur did not provide data on q_f and E_f , q_f and E_f values are included in Tables 2a and 2b for completeness and were calculated with Eqs. 2 and 10 based on the Δ and ℓ_f values reported by McArthur (1990).

The data reported by McArthur shows that ℓ_f varies between a minimum of 274.0 (data set 1) and a maximum of 435.0 W/m² (data set 3), Table 2a. The final air temperature (which is referred to as surface temperature by McArthur) ranges between 290.6 and 318.0°K, while estimates of Δ vary from 0.0796 (data set 1) to 0.442KPa/°K (data set 3). Furthermore, a cursory look at Tables 2a and 2b show that for each data set the variable estimates obtained with the four models are practically identical. Thus, subsequent discussion on the variable estimates will make no distinction between outputs from alternative models.

Accordingly, it can be observed from Tables 2a and 2b that estimates of ℓ_f obtained with the models, overall, vary between a minimum of about 277.1W/m² (data set 1) and a maximum of 435.6W/m² (data set 3) and those of E_f ranges from 9.8 to 15.4mm. On the other hand, estimates of T_a vary between a minimum of 291.4W/m² (data set 1) and a maximum of

Table 2a. Comparison of variable estimates computed with each of models 1 and 2 with those presented by McArthur (1990), McArthur's data sets

Variable name	Unit	Model outputs								
		McArthur's results			Model 1			Model 2		
		Data set 1	Data set 2	Data set 3	Data set 1	Data set 2	Data set 3	Data set 1	Data set 2	Data set 3
Latent heat flux, ℓ_f	W/m ²	274.0	372.0	435.0	277.1	373.0	435.6	277.1	373.0	435.6
Sensible heat flux, $q_f^{(*)}$	W/m ²	226.6	128.4	65.0	222.9	127.0	64.4	222.9	127.0	64.4
Evaporation flux, $E_f^{(*)}$	mm/day	9.7	13.1	15.3	9.8	13.1	15.4	9.8	13.1	15.4
Final air temperature, T_a	°K	290.6	303.0	318.0	291.4	303.5	318.3	291.4	303.5	318.3
Saturation vapor pressure related slope parameter, Δ	KPa/°K	0.0796	0.191	0.442	0.0821	0.1938	0.4465	0.0821	0.1938	0.4465
Percent absolute residuals ^(***)										
Variable name	Unit	-	between McArthur's results and model 1			between McArthur's results and model 2			-	
			Min	Max	Av	Min	Max	Av		
Latent heat flux, ℓ_f	% ^(**)	-	0.140	1.146	0.515	0.140	1.146	0.515		
Sensible heat flux, $q_f^{(*)}$	%	-	0.876	1.672	1.204	0.876	1.671	1.204		
Evaporation flux, $E_f^{(*)}$	%	-	0.140	1.146	0.515	0.140	1.151	0.517		
Final air temperature, T_a	%	-	0.101	0.281	0.182	0.101	0.281	0.182		
Saturation vapor pressure related slope parameter, Δ	%	-	1.016	3.110	1.858	1.016	3.109	1.858		

(*) McArthur (1990) did not provide estimates of sensible heat and evaporation fluxes. The values presented here were calculated, for completeness, with Eq. 2 and 10 based on the Δ values presented by McArthur.

(**) For each variable and data set combination, percent residuals were calculated with:

$$\text{absolute residual} = \frac{|\text{variable estimate with model 1 or 2} - \text{variable estimate by McArthur}|}{\text{variable estimate by McArthur}} \cdot 100$$

(***) Min, Max, and Av are the minimum, maximum, and average percent absolute residuals, respectively

Table 2b. Comparison of variable estimates computed with each of model 3 and 4 with those presented by McArthur (1990), McArthur's data sets

Variable name	Unit	Model outputs									
		McArthur's results			Model 3			Model 4			
		Data set 1	Data set 2	Data set 3	Data set 1	Data set 2	Data set 3	Data set 1	Data set 2	Data set 3	
Latent heat flux, ℓ_f	W/m ²	274.0	372.0	435.0	277.1	373.0	435.6	277.2	372.9	435.6	
Sensible heat flux, $q_f^{(*)}$	W/m ²	226.6	128.4	65.0	222.9	127.0	64.4	222.8	127.1	64.4	
Evaporation flux, $E_f^{(*)}$	mm/day	9.7	13.1	15.3	9.8	13.2	15.4	9.8	13.1	15.4	
Final air temperature, T_a	°K	290.6	303.0	318.0	291.4	303.5	318.3	291.4	303.5	318.3	
Saturation vapor pressure related slope parameter, Δ	KPa/°K	0.0796	0.191	0.442	0.0821	0.1938	0.4465	0.0821	0.1938	0.4465	
Percent absolute residuals ^(***)											
Variable name	Unit	-	between McArthur's results and model 3			between McArthur's results and model 4			Min	Max	Av
			Min	Max	Av	Min	Max	Av			
Latent heat flux, ℓ_f	% ^(**)	-				0.140	1.146	0.515	0.134	1.177	0.519
Sensible heat flux, $q_f^{(*)}$	%	-				0.876	1.671	1.204	0.842	1.705	1.193
Evaporation flux, $E_f^{(*)}$	%	-				0.140	1.146	0.515	0.134	1.177	0.519
Final air temperature, T_a	%	-				0.101	0.281	0.182	0.102	0.279	0.182
Saturation vapor pressure related slope parameter, Δ	%	-				1.016	3.109	1.858	1.020	3.087	1.855

(*) McArthur (1990) did not provide estimates of sensible heat and evaporation fluxes. The values presented here were calculated, for completeness, with Eq. 2 and 10 based on the Δ values presented by McArthur.

(**) For each variable and data set combination, percent residuals were calculated with

$$\text{absolute residual} = \frac{|\text{variable estimate with model 1 or 2} - \text{variable estimate by McArthur}|}{\text{variable estimate by McArthur}} 100$$

(***) Min, Max, and Av are the minimum, maximum, and average percent absolute residuals, respectively

318.3W/m² (data set 3) and Δ ranges from 0.0821 (data set 1) to 0.4465KPa/°K (data set 3). By comparison, sensible heat, q_f , vary between a minimum of 64.4W/m² (data set 3) and a maximum of 222.9W/m² (data set 1).

A close examination of the data in the outputs sections of Tables 2a and 2b shows that estimates of each variable, obtained with the models and those reported by McArthur, increase with the measured temperature. In other words, for a given model, estimate of a variable obtained for data set 1 is less than that for data set 2 and the estimate for data set 2 is less than that for data set 3 (Tables 2a and 2b). There, however, is an exception to this observation and it relates to the trend in the variability of sensible heat with the measured temperature. Unlike the other variables, sensible heat decreases as the measured air temperature increases. Given that the measured air temperature is the only input that is variable in the McArthur's data sets, it can be readily observed that the existence of distinct variability trends in relation to measured temperature is consistent with physical reasoning.

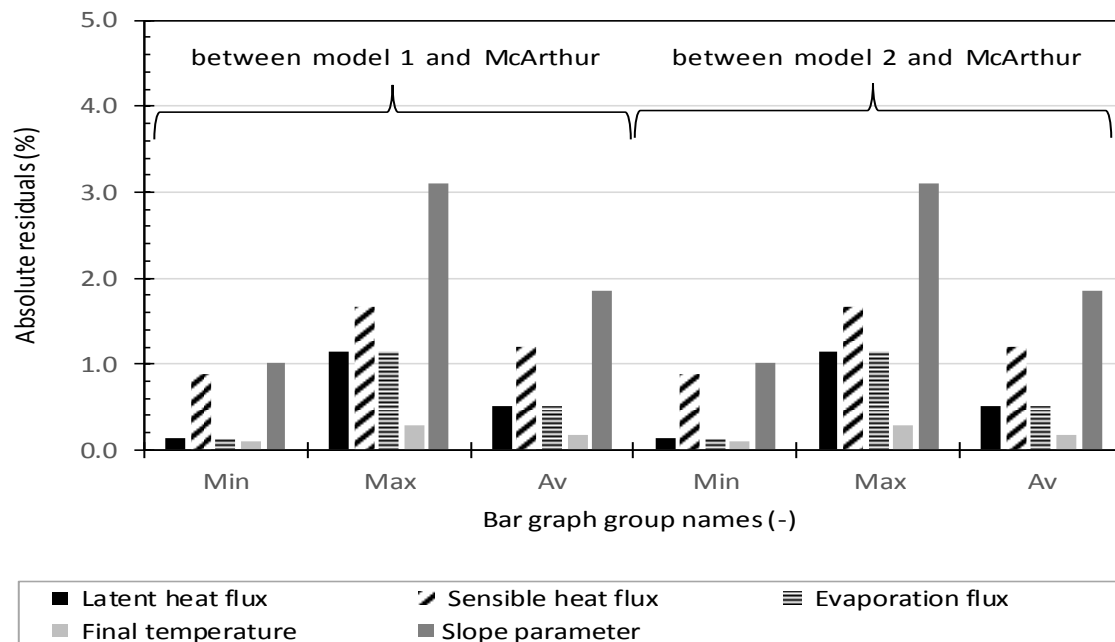
A descriptive summary of the percent absolute residuals between variable estimates obtained with each of the models and those reported by McArthur are given in Tables 2a and 2b. However, in order to allow for a more direct visual comparison the residuals are also plotted in grouped bar charts (Figure 1a for models 1 and 2) and (Figure 1b for models 3 and 4). A closer look at Figures 1a and 1b (and Tables 2a and 2b) shows that the maximum absolute residual between the variable estimates obtained with the alternative models and those of McArthur occur for Δ and it is about 3.1% and the average is 1.9%. The variable with the second largest residuals is the sensible heat flux, with a maximum of about 1.7%, the average value being 1.2%. On the other hand, the smallest residuals between the parameter estimates obtained with the proposed models and those of McArthur was obtained for the final air temperature, the maximum being

about 0.3% and the average 0.2%. The latent heat and evaporation fluxes fall somewhere in between with a maximum absolute residual of about 1.2% and an average value of 0.5%.

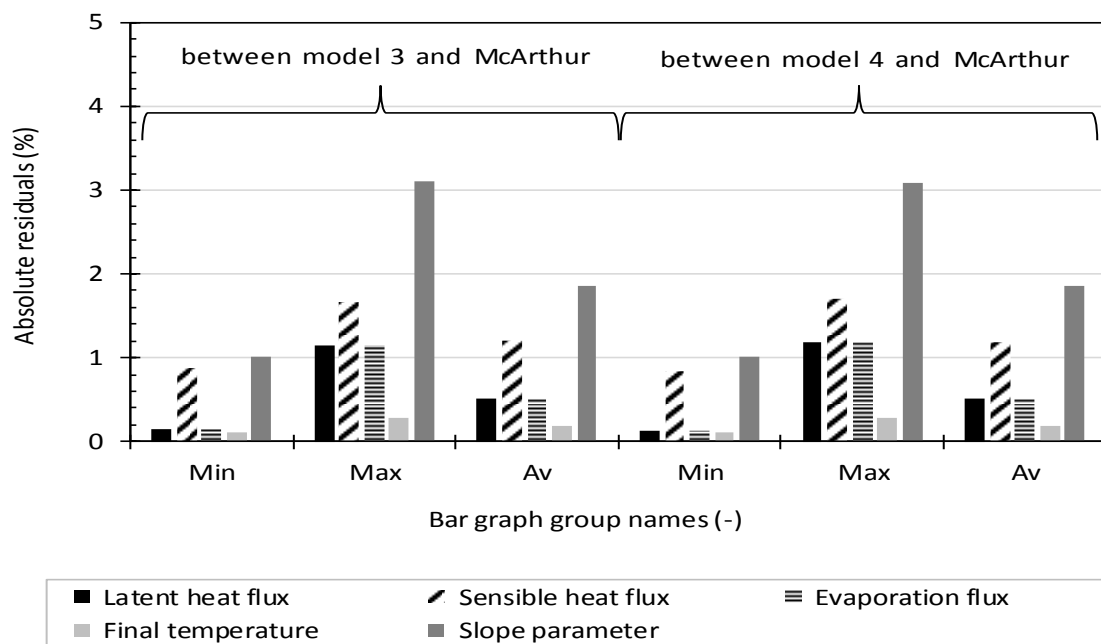
Overall, the percent absolute residuals summarized in Figures 1a and 1b and Tables 2a and 2b suggest that the variable estimates obtained with the proposed models and those reported by McArthur are generally close. Estimates for T_a , and the more important variables of latent heat flux and evaporation flux, are all on the average within about 0.5% of those reported by McArthur. However, the average residuals for Δ and q_f are greater than 1.0% (the maximum being 3.1%), hence slightly larger than what could be considered sufficiently small when comparing outputs of numerical solutions. Perhaps, differences in the programmatic implementation of the models (particularly model 1) in the current study and the specifics of (what appears to be a manual) calculation procedure used by McArthur (1990) may account for the slightly larger differences than would be desirable. Given that models 1, 2, 3, and 4 produce essentially identical outputs (Table 2), all the models will be considered further.

4.4.3. Comparison of models and model selection

The purpose of this evaluation is to ascertain, based on additional data sets, if estimates obtained with any one of the models show appreciable deviation from the others and if so, why? If not, the alternative models will be compared based additional criteria of numerical robustness, efficiency, and simplicity and the model with a better numerical attributes would be selected for further consideration.



(a)



(b)

Figure 1. Percent absolute residuals between variable estimates obtained with (a) each of models 1 and 2 and those presented by McArthur (1990) and (b) each of models 3 and 4 and those reported by McArthur

4.4.3.a. Comparison of the outputs of the alternative models based on additional data

Additional data, consisting of data sets 4 to 7 (Table 1), are used here to further assess if models 1, 2, 3, and 4 produce sufficiently close outputs over a wider range of evaporation conditions. Such an evaluation was conducted by comparing the values computed with each of models 2, 3, and 4 with those of model 1 for all the seven data sets. Accordingly, variable estimates computed with each of the alternative models and the corresponding percent absolute residuals for the McArthur and the additional data sets are presented in Tables 3a and 3b, respectively. Note that, for convenience, the model outputs section of Table 3a simply replicates the data presented in Tables 2a and 2b (i.e., variable estimates obtained with models 1, 2, 3, and 4 for the McArthur data sets) in a different format. The percent absolute residuals section of Table 3a, on the other hand, presents new data. The ranges, averages, and general trends inherent in the variable estimates of data sets 1, 2, and 3, Table 3a, were already described earlier in relation to the discussion in Tables 2a and 2b. Thus, the reader is referred to those sections of the chapter, if necessary.

In contrast to the variability trends observed for the McArthur data sets, no distinct variability trend, that holds across all the data sets, can be discerned in the variable estimates obtained with the alternative models. This is likely related to the fact that the input parameters that characterize the evaporation processes (consisting of vapor pressure deficit, measured air temperature, external heat flux, and aerodynamic resistance, and bulk surface resistance) do not exhibit a distinct variability pattern from one data set to another. A comparison of the variable estimates obtained with the alternative methods, will now be presented, based on the percent absolute residuals data (Tables 3a and 3b).

Table 3a. Comparison of variable estimates computed with each of models 2, 3, and 4 with those obtained using model 1, McArthur's data sets

Variable name	Unit	Model outputs											
		Model 1			Model 2			Model 3			Model 4		
		Data set 1	Data set 2	Data set 3	Data set 1	Data set 2	Data set 3	Data set 1	Data set 2	Data set 3	Data set 1	Data set 2	Data set 3
Latent heat flux, ℓ_f	W/m ²	277.1	373.0	435.6	277.1	373.0	435.6	277.1	373.0	435.6	277.2	372.9	435.6
Sensible heat flux, $q_f^{(*)}$	W/m ²	222.9	127.0	64.4	222.9	127.0	64.4	222.9	127.0	64.4	222.8	127.1	64.4
Evaporation flux, $E_f^{(*)}$	mm/day	9.8	13.1	15.4	9.8	13.1	15.4	9.8	13.1	15.4	9.8	13.1	15.4
Final air temperature, T_a	°K	291.4	303.5	318.3	291.4	303.5	318.3	291.4	303.5	318.3	291.4	303.5	318.3
Saturation vapor pressure related slope parameter, Δ	KPa/°K	0.0821	0.1938	0.4465	0.0821	0.1938	0.4465	0.0821	0.1938	0.4465	0.0821	0.1938	0.4465
Percent absolute residuals between estimates obtained with model 1 and those calculated with model 2, 3, and 4 ^(**)													
Variable name	Unit	-											
			Methods 1 and 2			Methods 1 and 3			Methods 1 and 4				
			Min	Max	Av	Min	Max	Av	Min	Max	Av		
Latent heat flux, ℓ_f	(%) ^(*)		0.0000	0.0005	0.0002	0.0000	0.0005	0.0002	0.006	0.030	0.016		
Sensible heat flux, $q_f^{(*)}$	(%)	-	0.0001	0.0006	0.0004	0.0000	0.0006	0.0003	0.033	0.034	0.033		
Evaporation flux, $E_f^{(*)}$	(%)	-	0.0000	0.0046	0.0016	0.0000	0.0005	0.0002	0.006	0.030	0.016		
Final air temperature, T_a	(%)	-	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.001	0.002	0.001		
Saturation vapor pressure related slope parameter, Δ	(%)	-	0.0001	0.0011	0.0006	0.0000	0.0011	0.0005	0.004	0.022	0.012		

(*) For each variable and data set combination percent absolute residuals were calculated with

$$\text{absolute residual} = \frac{|\text{variable estimate with model 2, 3, or 4} - \text{variable estimate with model 1}|}{\text{variable estimate with model 1}} 100$$

(**) Min, Max, Av are the minimum, maximum, and average percent absolute residuals, respectively.

Table 3b. Comparison of variable estimates computed with each of models 2, 3, and 4 with those obtained using model 1, additional data sets

Variable name	Unit	Model outputs															
		Model 1				Model 2				Model 3				Model 4			
		Data set 4	Data set 5	Data set 6	Data set 7	Data set 4	Data set 5	Data set 6	Data set 7	Data set 4	Data set 5	Data set 6	Data set 7	Data set 4	Data set 5	Data set 6	Data set 7
Latent heat flux, ℓ_f	W/m ²	191.3	339.8	520.5	683.3	191.3	339.8	520.5	683.3	191.3	339.8	520.5	683.3	191.2	339.8	520.5	683.2
Sensible heat flux, $q_f^{(*)}$	W/m ²	108.7	80.2	129.5	-283.3	108.7	80.2	129.5	-283.3	108.7	80.2	129.5	-283.3	108.8	80.2	129.5	-283.2
Evaporation flux, $E_f^{(*)}$	mm/day	6.7	12.0	18.3	24.1	6.7	12.0	18.3	24.1	6.7	12.0	18.3	24.1	6.7	12.0	18.3	24.1
Final air temperature, T_a	°K	286.8	314.5	312.5	286.4	286.8	314.5	312.5	286.4	286.8	314.5	312.5	286.4	286.9	314.5	312.5	286.4
Saturation vapor pressure related slope parameter, Δ	KPa/°K	0.0805	0.3265	0.3478	0.1217	0.0805	0.3265	0.3478	0.1217	0.0805	0.3265	0.3478	0.1217	0.0805	0.3265	0.3478	0.1217
Percent absolute residuals between estimates obtained with model 1 and those calculated with model 2, 3, and 4 (**)																	
		Model 1 and 2				Approaches 1 and 3				Approaches 1 and 4							
		Min	Max	Av		Min	Max	Av		Min	Max	Av		Min	Max	Av	
Latent heat flux, ℓ_f	(%) ^(*)	-				0.0000	0.0002	0.0001		0.0000	0.0004	0.0002		0.0059	0.0541	0.0206	
Sensible heat flux, $q_f^{(*)}$	(%)	-				0.0000	0.0004	0.0002		0.0001	0.0007	0.0003		0.0218	0.0915	0.0437	
Evaporation flux, $E_f^{(*)}$	(%)	-				0.0000	0.0002	0.0001		0.0000	0.0004	0.0002		0.0059	0.0541	0.0206	
Final air temperature, T_a	(%)	-				0.0000	0.0002	0.0001		0.0000	0.0000	0.0000		0.0003	0.0028	0.0012	
Saturation vapor pressure related slope parameter, Δ	(%)	-				0.0000	0.0007	0.0003		0.0001	0.0012	0.0006		0.0025	0.0271	0.0107	

(*) For each variable and data set combination percent absolute residuals were calculated with:

$$\text{absolute residual} = \frac{|\text{variable estimate with model 2, 3, or 4} - \text{variable estimate with model 1}|}{\text{variable estimate with model 1}} 100$$

(**) Min, Max, Av are the minimum, maximum, and average percent absolute residuals, respectively.

A quick look at the absolute residuals section of Tables 3a and 3b suggests that the variable estimates obtained with the models are essentially identical. However, in order to allow for a more direct visual comparison, a summary of the combined residuals data covering all the data sets are presented in a group bar chart (Figure 2). As can be noted from Figure 2 (and also from Tables 3a and 3b), the percent absolute residuals between the variable estimates obtained with model 2 and those of model 1 and between model 3 and those of model 1 are within 0.0046% of those computed with model 1. The maximum percent absolute residual between variable estimates obtained with model 4 and 1 is 0.092% of those computed with model 1 and it is associated with estimates of sensible heat flux. Furthermore, Figure 2 shows that the residuals for all the other variables and across all the data sets are less than 0.06%.

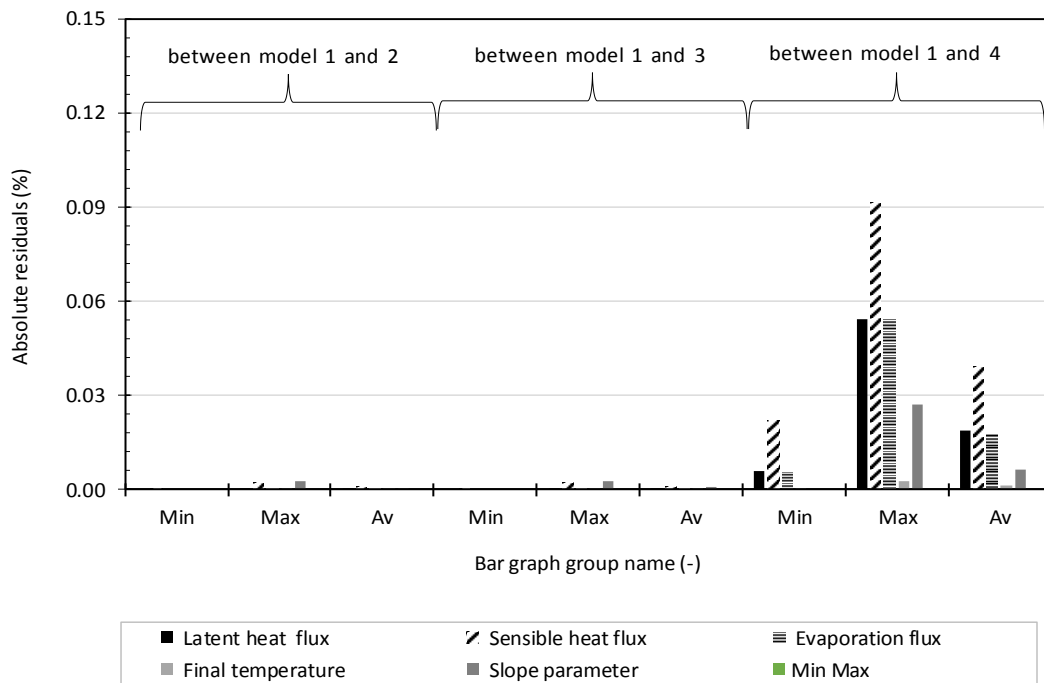


Figure 2. Percent absolute residuals between variable estimates computed with each of models 2, 3, and 4 and those obtained with model 1 (combined McArthur and additional data sets)

The fact that the variable estimates obtained with the proposed models are essentially identical, for the additional data sets as well, suggests that these algorithms can be considered equally acceptable.

4.4.3.b. Numerical robustness, efficiency, and simplicity

The results show that each of the numerical algorithms, presented here, were able to solve all the evaporation problems (i.e., data sets) considered. This implies that all the models presented here (i.e., model 1, 2, 3, and 4) are equally robust, although with varying degrees of efficiency. A close look at the iteration data shows that model 1 required 5 to 7 iterations to converge to the solutions for all the problems considered here, while model 2 converged to the solution within 2 to 4 iterations. Model 3, on the other hand, obtained the solution to all the seven data sets in 2 iterations. By contrast, model 4 required by far the largest number of iterations, ranging between 169 and 1033 to obtain the solutions. While these differences in apparent numerical efficiency may be significant in terms of CPU time, since the evaporation problem concerns the solution of small system of equations (at most three equations in three variables), the actual runtimes for each of these models are nearly indistinguishable and are about a few seconds in a regular PC.

The preceding observations suggest that from the perspectives of accuracy and numerical robustness and efficiency (in the sense of required model runtime), each of the models represent a comparable alternative to any of the other alternatives for estimating evaporation from a cropped field. However, the fact that model 1 is based on function evaluation implies that its implementation does not require any specialized numerical method. By comparison, models 2, 3, and 4 use Newton iteration method to solve a minimum of 1 and a maximum of 3 nonlinear equations with 1 to 3 variables, respectively. The implication is that model 1 is simpler than the

other models, both conceptually and in terms of its numerical formulation and programmatic implementation, which is an advantage. Thus, model 1 is considered here for further use.

The fact that the solution obtained for the complete set of equations is practically identical with those computed with the reduced form, evidently validates the simplified approach for solving Eqs. 1 to 4, which was justified based on the observed lack of strong coupling between Eqs. 1 and 2, on the one hand, and Eqs. 3 and 4, on other.

4.5. Comparison of model 1 and the conventional model

Estimates of ℓ_f , q_f , E_f , T_a , and Δ computed with model 1 are compared with those obtained with the conventional model for the seven data sets presented in Table 1. The goal is to assess the advantages and limitations of model 1 relative to that of the conventional approach.

4.5.1.a. Latent heat flux

Table 4 presents the variables estimates obtained with model 1 and the conventional model for all the data sets. Note that the data in the outputs section of model 1 simply replicates those presented in Tables 3a and 3b for convenience. As can be noted from the descriptive summary section of Table 4, the latent heat flux computed with model 1 varies between a minimum of 191.3W/m^2 (data set 4) and a maximum of 683.3W/m^2 (data set 7), the average value being 402.9W/m^2 . By comparison estimates of ℓ_f obtained with the conventional method range from 175.1W/m^2 (data et 4) to 654.1W/m^2 (data set 7) and the mean value is 376.0W/m^2 . Furthermore, evaporation flux obtained with model 1 vary between 6.7mm/day (data set 4) and 24.1mm/day (data set 7), with an average value of 14.2mm/day . The conventional approach predicts slightly

Table 4. Comparison of variables estimates computed with model 1 and the conventional model

		Model outputs													
Variable name	Unit	Model 1							Conventional model						
		McArthur's data sets				Additional data sets			McArthur's data sets				Additional data sets		
		Data set 1	Data set 2	Data set 3	Data set 4	Data set 5	Data set 6	Data set 7	Data set 1	Data set 2	Data set 3	Data set 4	Data set 5	Data set 6	Data set 7
Latent heat flux, ℓ_f	W/m ²	277.1	373.0	435.6	191.3	339.8	520.5	683.3	201.4	343.7	428.4	175.1	320.2	509.5	654.1
Sensible heat flux, $q_f^{(*)}$	W/m ²	222.9	127.0	64.4	108.7	80.2	129.5	-283.3	298.6	156.3	71.6	124.9	99.8	140.5	-254.1
Evaporation flux, $E_f^{(*)}$	mm/day	9.8	13.1	15.4	6.7	12.0	18.3	24.1	7.1	12.1	15.1	6.2	11.3	18.0	23.1
Final air temperature, T_a	°K	291.4	303.5	318.3	286.8	314.5	312.5	286.4	297.7	305.9	318.9	288.2	317.3	312.9	287.1
Saturation vapor pressure related slope parameter, Δ	KPa/°K	0.0821	0.1938	0.4465	0.0805	0.3265	0.3478	0.1217	0.0445	0.1452	0.3946	0.0610	0.2442	0.3119	0.1452
Modified psychrometer constant, $\gamma^{(*)}$	KPa/°K	-	-	-	0.0697	0.0925	0.1107	0.0816	-	-	-	0.0697	0.0925	0.1107	0.0816
Aerodynamic resistance, $r_a^{(*)}$	s/m	-	-	-	98.4	173.1	42.2	28.2	-	-	-	98.4	173.1	42.2	28.2
Variable name	Unit	Descriptive summary of computed variables (**)							Summary of residuals between variable estimates obtained with model 1 and the conventional model (%) (***)						
		Model 1			Conventional model				Min			Max		Av	
		Min	Max	Av	Min	Max	Av								
Latent heat flux, ℓ_f	W/m ²	191.3	683.3	402.9	175.1	654.1	376.0	1.66			27.32		8.21		
Sensible heat flux, $q_f^{(*)}$	W/m ²	-283.3	222.9	145.1	-254.1	298.6	163.7	8.53			33.98		18.06		
Evaporation flux, $E_f^{(*)}$	mm/day	6.7	24.1	14.2	6.2	23.1	13.3	1.67			27.33		8.21		
Final air temperature, T_a	°K	286.4	318.3	301.9	287.1	318.9	304.0	0.12			2.15		0.69		
Saturation vapor pressure related slope parameter, Δ	KPa/°K	0.0805	0.4465	0.2284	0.0445	0.3946	0.1924	10.32			45.77		23.07		

(*) Modified psychrometer constant is not given for the McArthur data sets, because it is equal to the psychrometer constant, which was specified at the input (Table 1). The aerodynamic resistance as well is not given at output for McArthur's data sets, because it was specified at the input

(**) Min, Max, Av are the minimum, maximum, and average percent absolute residuals, respectively.

(***) For each variable and data set combination, percent absolute residuals were calculated with:

$$\text{absolute residual} = \frac{|\text{variable estimate with model 1} - \text{variable estimate with the conventional model}|}{\text{variable estimate with model 1}} 100$$

less evaporation fluxes, E_f , than that of model 1, which vary between 6.2mm (data set 4) and 23.1mm/day (data set 7), with an average of 13.3mm/day.

A close look at Table 4 shows that the minimum and maximum of the computed latent heat (or evaporation) flux obtained with both models are observed in data sets 4 and 7, respectively. The low evaporation flux associated with data set 4 can be explained by the fact that data set 4 is characterized by a relatively small external heat flux (300W/m^2), low vapor pressure deficit (0.373KPa), low air temperature (278°K or 5°C), and a relatively large aerodynamic resistance of 98.4s/m . On the other hand, the external heat flux for data set 7, which is 400W/m^2 , is about 85% of the average across all data sets and the measured air temperature of 293°K is not on the high side. Thus, the likely explanatory factor for the very large latent heat and evaporation fluxes are the very high vapor pressure deficit (of 2.1KPa) and to a degree the small aerodynamic resistance of 28.2s/m . Note that the low aerodynamic resistance is at least in part attributable to the relatively high wind speed of 3.0m/s (Table 1). This suggests that in data set 7, the evaporation process was likely dominated by the adiabatic cooling process.

Upon closer examination, the latent heat flux for data set 7 show that the gain in latent heat, in the adiabatic cooling process, expressed in terms of equivalent latent heat flux is 443.9W/m^2 . By comparison, the fraction of the external heat flux, that led to further increases in latent heat content of the air in the diabatic process is 239.4W/m^2 , which is about 53.9% of that associated with the adiabatic component. This, evidently, shows that the evaporation process in data set 7 is dominated by the adiabatic cooling component, driven mainly by the high vapor pressure deficit and to a degree by the low aerodynamic resistance.

4.5.1.b. *Sensible heat flux*

The sensible heat flux computed with model 1 varies between a minimum of -283.3W/m^2 (data set 7) and a maximum of 222.9W/m^2 (data set 1), with an average value of 145.1W/m^2 (note that the average q_f is calculated here based on the absolute value of the computed sensible heat fluxes). On the other hand, q_f obtained with the conventional method ranges from -254.1W/m^2 (data set 7) to 298.6W/m^2 (data set 1) and the average is 163.7W/m^2 (Table 4). The negative algebraic sign of the minimum sensible heat flux in data set 7 indicates that there was a net decline in the sensible heat content of the air during evaporation. However, whether the decrease in the sensible heat content of the air represents not only a net reduction (associated with the adiabatic cooling process that is not fully compensated in the diabatic process), but also a net outflow of heat from the air to the exchange surface depends on the algebraic sign of the diabatic component of the sensible heat flux.

A close look at the sensible heat flux data of data set 7, computed with model 1, reveals that the equivalent heat flux (or the steady rate with which the sensible heat content of the air decreased) in the adiabatic cooling process is -443.9W/m^2 . On the other hand, the fraction of the external heat flux that accounts for the change in the sensible heat content of the air in the diabatic process is 160.6W/m^2 . The positive algebraic sign of the sensible heat flux associated with the diabatic component shows that it represents a transfer of heat from the exchange surface to the air and hence an increase in the sensible heat content of the air (note that this observation could have also been deduced from the algebraic sign of the external heat flux for data set 7, which is 400W/m^2). These data show that the negative algebraic sign of the net sensible heat flux for the entire evaporation process (-283.3W/m^2 , Table 4), which is equal to the algebraic sum of

the adiabatic and diabatic components of the sensible heat flux, is not caused by heat transfer from the air to the exchange surface (Table 4). Instead, it is related to the decrement in the sensible heat content of the air in the adiabatic process through conversion to latent heat (at the rate of -443.9W/m^2) that was not fully compensated by the increment that occurred in the diabatic process (160.6W/m^2), leading to a net overall decline in the sensible heat content of the ambient air.

4.5.1.c. Final (equilibrium) air temperature and slope parameter related to the saturation vapor pressure function

Estimates of the final air temperature obtained with model 1 vary between a minimum of 286.4°K and a maximum of 318.3°K , with an average value of 301.9°K (Table 4). The surface temperature (which is equivalent to the final air temperature in model 1) calculated with the conventional model ranges between 287.1 and 318.9°K and the average is 304.0°K . Furthermore, estimates of Δ obtained with model 1 ranges from 0.0805 to $0.4465\text{KPa}/^\circ\text{K}$ and the mean is $0.2884\text{KPa}/^\circ\text{K}$. By comparison, the Δ value calculated with the conventional model vary between 0.0445 and $0.3946\text{KPa}/^\circ\text{K}$ and the average is $0.1924\text{KPa}/^\circ\text{K}$. A close look at the data shows that T_a and Δ estimates computed with both models show strong positive correlation with the measured air temperature, T_0 (correlation coefficient exceeding 0.9) for data sets 1 to 6 (Table 1). This suggests that T_0 can be considered a good indicator parameter of the relative magnitudes of the T_a and Δ variables.

4.5.1.d. Percent absolute residuals between variable estimates obtained with model 1 and the conventional model

A summary of the overall percent absolute residuals between the variable estimates obtained with model 1 and those of the conventional model is presented in the residuals section of Table 4. In addition, however, the minimum, maximum, and average percent absolute residuals between variable estimates computed with model 1 and those obtained with the conventional model are depicted in a grouped bar chart (Figure 3). As can be noted from Figure 3 and Table 4, with a maximum percent absolute residual of about 45.8% and a mean residual of 23.1%, the largest residual between the variables computed with the conventional model and model 1 is associated with estimates of Δ .

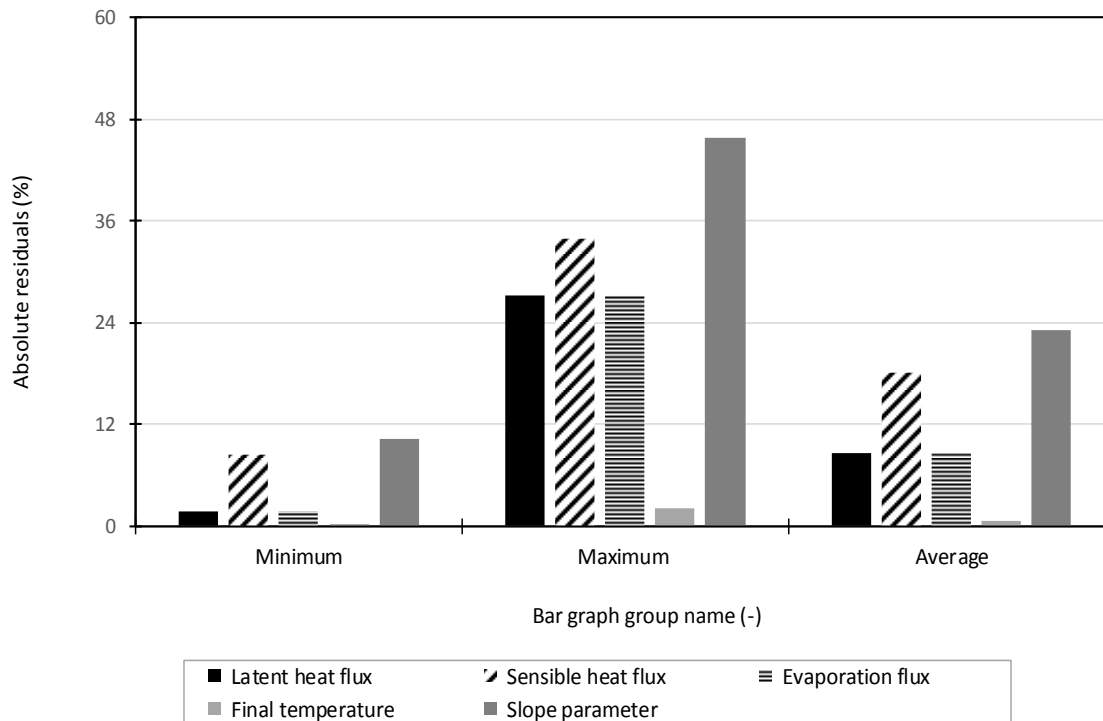


Figure 3. Percent absolute residuals between variable estimates computed with model 1 and those obtained with the conventional model (combined McArthur and additional data sets)

By comparison, the smallest percent absolute residual is observed for the final air temperature, with a maximum value of about 2.2% and a mean of 0.7%. The results also show that the variable most affected by the difference in the approaches used to estimate Δ , in the models, is the sensible heat flux, which has a maximum residual of about 34.0% and an average value of 18.1%. Although the rather more important parameters of latent heat flux and evaporation flux have a relatively large maximum residual of about 27.3%, the average residual for both parameters is a much smaller amount of 8.2% (Table 4). These results suggest that differences in the methods used for estimating Δ have the maximum effect on sensible heat flux estimates, a negligible effect on estimates of T_a , and a limited effect on the estimates of latent heat and evaporation fluxes.

It is important to note here that both model 1 and the conventional model are based on simplifying assumptions and hence outputs of both models involve a level of approximation. The implication is that the accuracy of the conventional model and model 1 cannot be evaluated through direct intercomparison of the models. Such an assessment, instead, requires the variable estimates obtained with each model (i.e., the conventional model and model 1) to be compared with those of a more accurate model (likely a mechanistic simulation model) of evaporation from a cropped field, that couples vapor, heat, and momentum transfer and transport processes through the soil-crop-atmosphere continuum, taking into account wind induced advection and forced convection. Such an evaluation is not part of the current study, thus, assessment of the accuracy of the proposed model (model 1) relative to that of the conventional approach is not conducted here. However, considering evaporation flux (i.e., the single most important variable in so far as agricultural water management is concerned), the relatively large maximum percent absolute residual of 27.3% suggests that in some cases there can be significant difference between the

evaporation flux estimates of the two methods. This may justify the need for a follow up study aimed at determining the relative accuracy of the two models.

Generally, the average residual, instead of the maximum, is more representative of the difference between variable estimates computed with model 1 and the conventional model. Thus, the relatively small mean absolute residual (of 8.2%) for ℓ_f suggests that the margins of error in the ℓ_f (and hence E_f) estimates obtained with these models may not be significantly different. The implication is that, at least for a good set of applications, the use of both the conventional model and model 1 can be considered equally acceptable from the standpoint of accuracy. However, the conventional model is preferred, because of its simplicity. Nonetheless, it ought to be emphasized that the results presented here are of limited nature and hence a conclusive deduction on this may need to await a more comprehensive follow up study.

Chapter 5. Summary and conclusion

A review of Monteith's thermodynamic approach to the derivation of the Penman-Monteith equation is presented in Chapter 2. The derivation presented here, however, places emphasis on the basic assumptions and attempts to fill gaps, mainly in the form of providing extended discussion in places where the original discussion was concise. In addition, it focusses on some useful mathematical/physical interpretations of the terms and key parameters of the equation that the derivation reveals.

The thermodynamic conceptualization of evaporation introduces a perspective whereby evaporation is viewed from the vantage point of the changes it introduces into the thermodynamic properties of the air. Accordingly, the derivation of the Penman-Monteith

equation presented here describes evaporation as a physical process consisting of a pair of formal thermodynamic subprocesses (namely, adiabatic cooling and diabatic heating) that leads to changes in the energy states of the ambient air in a way that is readily quantifiable. The Penman-Monteith equation is derived in two steps. As an initial approximation, a form of the equation that models evaporation into a stationary ambient air is developed based on the thermodynamic equations of state applied to a suitably defined system. Transfer (resistance) parameters are then introduced into the basic equations, in a subsequent step, accounting for the dynamic effects of wind-surface interactions (and the effects of crop physiological response to atmospheric conditions) on evaporation, leading to the Penman-Monteith equation. Although less compact than the conventional approach, the thermodynamic approach to the derivation of the Penman-Monteith equation has the advantage of being revealing of key assumptions and concepts that are generally implicit in the conventional approach.

The initial step of the thermodynamic formulation accentuates the notion that the Penman-Monteith equation is fundamentally a description of the process of vapor and heat transfer between a wet source/sink surface and a stationary ambient air. The subsequent step, on the other hand, emphasizes the fact that the convective transfer of vapor, heat, and momentum between the exchange surface and the ambient air along with crop stomatal effects on evaporation are taken into account in an approximate sense. The thermodynamic approach also shows that evaporation is a process essentially driven by energy (heat) supply and as such each term of the Penman-Monteith equation represents a separate heat source for evaporation, consisting of external heat for the diabatic process and sensible heat extant in the air at the start of evaporation for the adiabatic process.

Considering that the thermodynamic formulation is based on conceptual charts depicting the energy states of a suitably defined thermodynamic system, it readily reveals some useful interpretations of the mathematical/physical attributes of key parameters of the Penman-monteith equation, namely the slope parameter related to the saturation vapor pressure curve, Δ , the psychrometer constant, γ , and the modified psychrometer constant, γ^* . The results also suggest that the thermodynamic description of the terms of the Penman-Monteith equation as adiabatic and diabatic components might be a more precise interpretation of the physical meaning of the terms.

The derivation shows that the introduction of the air density parameter into the thermodynamic equations of state and hence into the Penman-Monteith equation, which resulted in heat contents of the air being expressed in a unit volume basis, presumes that the range of (air) temperature variation during evaporation, i.e., $[T_a - T_w]$, is sufficiently small for the density of the air to be adequately represented by an average value. Note that this constraint on the range of temperature variation is independent of the requirement that relate to the constancy assumption of the Δ parameter.

The derivation presented in Chapter 2 produces a small system of equations for computing vapor and heat flux from a source/sink surface to the ambient air and the change in air temperature during evaporation. However, another important physical process integral to evaporation and is inextricably coupled to the vapor and heat transfer processes is that of momentum transfer. Natural evaporation is generally conceptualized as a process driven mainly by near surface turbulent transfer and transport processes, caused by wind induced forced

convection coupled to soil-canopy complex effects. Thus, the momentum transfer equation is key to developing expressions for the resistance parameters to vapor and heat transfer as well (which are parameters of the system of equations derived in Chapter 2). Accordingly, the derivation of the equation for the resistance parameter to momentum transfer (or the aerodynamic resistance) and its subsequent adaptation for use in quantifying the resistance parameters to vapor and heat transfer, under a suitably defined atmospheric condition, is described in Chapter 3.

The equations derived in Chapter 2, consisting of the latent heat flux, sensible heat flux, final air temperature, and Δ equations cannot be solved directly, because the value of Δ is not known. Thus, the system of equations, which are referred here as the Penman-Monteith system of equations, are solved numerically. A description of the numerical solutions developed as part of the current study is presented in Chapter 4. Taking advantage of the mathematical structure of the equations, alternative numerical solutions ranging in complexity from those involving the solution of a subset of the system of equations to those solving a form of the complete set are developed and evaluated here. Accordingly, four alternative algorithms, referred to as model 1, 2, 3, and 4, are presented. While model 4 computes latent heat, ℓ_f , sensible heat, q_f , and final air temperature, T_a , through a simultaneous solution of a form of the complete set of equations; models 1, 2, and 3, on the other hand, use a two-step approach to determine the variables: ℓ_f , q_f , T_a , and Δ . The first phase uses different formulations/iteration methods to solve the final air temperature and the Δ equation for T_a and Δ . Then, in the second phase the Δ value so computed is substituted into the ℓ_f and q_f equations to determine ℓ_f and q_f , respectively.

Results of model verification showed that each of the alternative models produced outputs practically identical and also in close agreement with a reference solution. The average

absolute residuals between the predictions of the models presented here and those of the reference solution vary between about 0.2% observed for T_a and a maximum of 1.9% for Δ . In addition, comparison of the alternative models based on the criteria of numerical efficiency and robustness suggests that each model represents a comparable alternative, to any of the other models, for estimating evaporation. However, owing to its simplicity, model 1 is recommended for further use.

The models developed in the current study, including model 1, compute Δ as part of the numerical solution. A more widely used approach described here as the conventional model, on the other hand, determines Δ independently (by setting it equal to the slope of the saturation vapor pressure curve at the measured air temperature), then the Δ value obtained as such is used to calculate ℓ_f , q_f , and T_a with the respective equations directly. A comparison of the outputs of the conventional model with those of model 1 was conducted, here, based on a total of seven hypothetical data sets covering a wide range of natural evaporation conditions. The results suggest that differences in the methods used for estimating Δ have the maximum effect on sensible heat flux estimates (where the mean absolute residual is 18.1%), a negligible effect on estimates of final air temperature (with an average residual of 0.7%), and a limited effect on the estimates of latent heat and evaporation fluxes, in which the mean residual is 8.2%.

Both model 1 and the conventional model involve a level of approximation in the determination of Δ , thus a direct comparison of the two models cannot provide answer to the question: which model is more accurate? Consequently, the current study does not try to answer that question. However, considering the latent heat flux, ℓ_f (the most important of the evaporation variables in agricultural water management applications), the fact that the average residual over all the data sets is only 8.2% suggests that the margins of error for the ℓ_f estimates obtained with

these models may not be significantly different. This implies that, at least for a good set of applications, the use of both the conventional model and model 1 can be considered equally valid from the standpoint of accuracy. However, the conventional model is preferred, because of its simplicity. Nonetheless, it ought to be emphasized that the study presented here is based on a limited data and hence a conclusive deduction needs to await a more comprehensive follow up study.

Chapter 6. References

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